PROCESS QUALITY MODELS AND THEIR
APPLICATION TO OPTIMIZING QUALITY CONTROL PROCEDURES

A THESIS

Presented to

The Faculty of the Graduate Division

by

Dev Kumar Ray

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Industrial Engineering

Georgia Institute of Technology

June, 1963

Ray

PROCESS QUALITY MODELS AND THEIR

APPLICATION TO OPTIMIZING QUALITY CONTROL PROCEDURES

Approved:

Date approved by Chairman: June 6, 1963

ACKNOWLEDGMENTS

The author expresses grateful appreciation to all who have made this study possible.

In particular, he is grateful to his advisor, Dr. Joseph Moder, whose continuous guidance is responsible for whatever value this thesis may have. He also extends thanks to Drs. Harrison Wadsworth and Eric Immel for their constructive suggestions.

However, the author is solely responsible for any error that may be present in the thesis.

TABLE OF CONTENTS

SUMMARY

The objective of this study is to present a framework for the classification and development of optimization techniques for the design of statistical quality control chart procedures. The criterion of optimality is the minimization of the long run average net quality losses of a process. Although only control charts for variables have been considered, the framework can be easily extended to incorporate control charts for attributes as well. Acceptance sampling plans have not been considered.

Three levels of sophistication in the design of a statistical quality control chart procedure have been defined. In Level One, past practice and rule-of-thumb bases are used to determine the parameters of the quality control procedure. Level Two is further refined by considering the important process characteristics for the design of the parameters of the control procedure. In Level Three, all the important economic and operating characteristics of the process are taken into account.

A comprehensive set of six process quality models is suggested in order to: 1) assist in the formulation of a specific model for a process quality problem, 2) serve as a means of classifying the existing literature on the optimizing techniques of quality control chart procedure, and 3) point out the areas where additional research may be directed.

The first three models are used for controlling only the process mean, while the fourth model is used for controlling only the process variability. The first two models describe basically deterministic behaviors of the process mean and the third model describes a time dependent, stochastic behavior of the process mean. When both the process mean and the process variability need to be controlled, one of the first three models is combined with the fourth model to yield Model V. While the random variables involved in the first five models are assumed to be normally distributed, the sixth model is used when the random variables in the process are known to be non-normal.

The survey of the literature indicates that a substantial amount of research is needed on some of the process quality models, especially Models II, III and VI. There is also an urgent need for the consideration of the economic factors in making Level Three designs. Moreover, the developments in the optimal design of quality control charts should be translated into simplified rules to make them more readily usable in industry.

CHAPTER I

INTRODUCTION

Inspection has been in existence from time immemorial, but statistical quality control, as it is known today, is of recent origin. Inspection and quality control, based on statistics, began in the early years of the twentieth century. The earliest works on industrial quality control were performed by a small group of people at the Western Electric Laboratories and the Bell Telephone Laboratories. E. C. Molina, of the Western Electric Company, is known for computing the tables of the Poisson distribution. The earliest papers dealing specifically with acceptance inspection were written by Coggins and Shewhart. The 1929 Dodge-Romig plan has remained virtually unaltered to this day and nearly all other plans have had considerable inspiration from it.

Spurred by the Second World War, the developments after 1941 were rapid. Various groups were organized, two being at Columbia University and at Stanford University. After the war, the security regulations were lifted and the Department of Defense issued various sampling plans which are widely used by industry today. During the past decade, an enormous volume of work has been done on statistical quality control. The American Society of Quality Control was born in 1946 and has been issuing its journals regularly ever since. In 1952, the Royal Statistical Society began the publication of the journal Applied Statistics. Statistical quality control has now spread to many countries and relevant societies exist in Europe and the Far East.

Quality Control and Engineering Economy

Although development of statistical quality control has been indirectly spurred by ultimate economic objectives, the plans have very seldom taken all the economic facets explicitly in consideration. Engineering economy and quality control often serve identical purposes, but conventional engineering economy techniques are often difficult to apply to matters of quality.

In the management of any productive enterprise, decisions of various kinds are required. Where technical considerations are involved in the alternative courses of action open to the decision maker, a study comparing specific economic estimates of the alternatives as well as a consideration of other factors, not readily expressable in economic terms, is called an engineering economy study. Engineering economy studies dealing with quality matters are often very difficult to carry out because of the difficulty of expressing in economic terms the probable effects of particular quality decisions, and due to the fact that the accounts of an enterprise usually do not identify the costs of spoilage and costs associated with selling defective items to customers. Perhaps for this reason, most quality decisions in practice seem to be made on an intuitive basis. During the last decade, however, there have been some sporadic efforts to state the principles of choice between quality control alternatives more explicitly and this study deals with some of these efforts.

Meaning of Quality

Before dealing with the optimization techniques of statistical quality control, it is necessary to define "quality" or, in other words, "product quality." The term "product quality" can very well be used since the conventional quality control tools are chiefly used in productive enterprises and not in service enterprises. According to Feigenbaum (16), product quality can be defined as "the composite product characteristics of engineering and manufacture that determine the degree to which the product in use will meet the expectations of the customer." The terms, such as reliability, serviceability, and maintainability, which are sometimes used as definitions of product quality, are individual characteristics which go to make up the composite product quality.

Product quality can be divided into two classes: First, the quality of design or the value inherent in the design and second, the quality of conformance to that design. Statistical quality control procedures deal chiefly with the quality of conformance and influence the quality of design only indirectly. The effect of the quality control procedures on the design of the product will be ignored in this study principally because of the indirect relationship between the two.

Costs of Quality

The total cost of a product decreases, or the value of the finished product increases with the increase in the quality of conformance, until an optimum level is reached. Beyond this optimum, when "perfectionism" sets in, the increased cost of achieving better quality more than offsets the greater market value of the finished product. Hence the basic purpose of quality control is to strike the optimum balance between the costs of quality and the value of quality for each product.

The costs of quality can be divided into the following categories:

1. Appraisal costs. These include the expenses for maintaining company quality levels by means of formal evaluations of product quality. Specifically included are incoming, process, and final inspection and testing.

2. Failure costs. These costs are caused by defective material and products that do not meet company quality specifications. Included are scrap, rework, all costs of complaint adjustments, and all costs for customer services including the loss of goodwill.

3. Prevention costs. These costs are incurred for the purpose of keeping defects from occurring in the first place. Prevention costs consist chiefly of quality control engineering costs.

The prevention costs category now amounts to perhaps 5 to 10 per cent of the total quality costs, which suggests that the quality control expenses are incurred the wrong way. It appears to be possible to cut the two major cost segments of a company's quality costs by means of much smaller increases in the prevention cost segment. These circumstances forecast a sharp increase in the importance of the prevention cost category, or in the emphasis on "in-process" statistical quality control methods.

Purpose of Quality Control

The optimization of quality control procedures is an endeavor to minimize the sum total of the three different quality costs. Of these three classes of costs, the failure costs are inherently the most difficult to evaluate. The greatest difficulty occurs in the consumers' goods industries, where the product goes to a great many different customers who make no formal acceptance tests. It is, therefore, often necessary to make certain assumptions regarding the cost parameters in order to take all the important factors into consideration.

Levels of Inspection

To minimize the total quality costs, it is sometimes economical to do no inspection at all, sometimes most economical to have 100 per cent inspection, and sometimes best to have a sampling inspection of some type. In view of the objective of maximum economy, certain conditions are evident that are favorable to each of these levels of inspection. Where a product is consistently satisfactory for the purpose intended, it is likely to be most economical to have no inspection whatever. Sometimes, however, concealed opportunities may exist in the process for reducing production costs and such opportunities may only be disclosed by regular sampling inspection using control charts. Low unit failure costs may also make it economical to do no inspection whatever. For example, where an unsatisfactory product is readily discovered and eliminated in a subsequent production operation, it may be cheaper to tolerate a moderate percentage of such product than to eliminate it by inspection.

Where a product is consistent in quality but nearly always contains a substantial percentage of unsatisfactory product, 100 per cent inspection may be the most economical alternative. Here the choice is likely to be between 100 per cent inspection and no inspection. The higher the percentage of unsatisfactory product produced and the higher the failure cost per unit of such product, the more favorable the conditions for 100 per cent inspection.

Sampling inspection is most economical when a product is usually good enough for no inspection to be more economical than 100 per cent inspection, and where the product is occasionally bad enough for 100 per cent inspection to be more economical. It should be recognized at this stage that sampling inspection schemes may possibly reduce the quality costs in two ways. One way is by the rectification or rejection of the relatively bad lots of product, thereby making the proportion of unsatisfactory product approved less than the proportion submitted. The other way is by reducing the proportion of unsatisfactory product submitted. Sampling inspection may improve product quality through the diagnosis of causes of quality troubles and through the exertion of effective pressure for process improvement.

Quality Losses

For most products there are many quality characteristics to be controlled. The quality losses are distributed among all these quality characteristics but fortunately they are not usually uniformly distributed. The losses are often maldistributed in such a way that a small percentage of the quality characteristics always contributes to a high percentage of the quality loss. This is fortunate because significant improvement can be secured at a minimum of investigation cost. The investigation plus the correction cost for any quality characteristic is substantial and should be undertaken only when the losses incurred are more than enough to justify a study for improvement. For most quality characteristics, the cost of investigation and correction exceeds the quality losses.

In this age of increasing competition between enterprises and the rise in precision of the products, all the above-mentioned factors denote a need for an increase in emphasis on minimizing quality costs. This is particularly true since quality costs frequently amount to as much as 10 to 12 per cent of the total cost. The nature of this increased emphasis is also significant. In the past, most of the efforts have been devoted to working out the details of applying the conventional charts to the production processes. The current trend towards automated processes, together with the use of high-speed computers in data processing, point towards the necessity of broadening the scope of the developments in the quality control field. Moreover, with the development of applied statistics and operations research, more sophisticated tools are being developed to cater to the needs of optimizing quality control procedures. The purpose of this study is to present a framework for the development of these more sophisticated procedures.

CHAPTER II

THE ROLE OF THE MATHEMATICAL MODEL IN THE DESIGN OF AN OPTIMAL QUALITY CONTROL PROCEDURE

Levels of Sophistication

There are several possible levels of sophistication in the design of a statistical control chart procedure. These can be broadly classified under three levels:

Level One. Application of the conventional $\bar{X}$ and R charts to give 3-sigma control limits, and using past practice or other rule-of-thumb bases to specify the sampling procedure.

Level Two. Adjustment, on a semi-quantitative basis, of the sampling procedure and the control-chart decision rules used in Level One, in view of certain characteristics of the process in question.

Level Three. Adjustment on a quantitative basis of the sampling procedure and the control-chart decision rules used in Level One, giving full consideration to all of the important economic and operating characteristics of the process in question.

This classification system will be described in more detail and will be illustrated by hypothetical examples.

First Level of Sophistication

The conventional control-chart and sampling procedures constitute the first level of sophistication. This applies equally well to control charts by variables and attributes. The sample size and the frequency of sampling are determined more or less intuitively.

Illustration. To illustrate the design of a control chart by variables, consider a measurable quality characteristic X. In order to control the process, both the average and the general variability of the process should be controlled, especially if $(USL - LSL)/\sigma$ is small, where USL and LSL are the Upper and Lower Specification Limits, respectively, and σ is the process standard deviation. The most convenient control charts to be employed for this purpose are $\bar{X}$ and R charts. The target value of $\bar{X}$ and R can be determined from past data and the 3-sigma control limits are installed in the usual manner.

The criteria for detecting lack of control may be:

1. a point outside of the control limits, or

2. two successive points falling outside warning lines but within the control limits where the warning lines may be set at $\bar{\bar{X}} \pm \frac{2\bar{R}}{d_2\sqrt{n}}$ and $\bar{R} \pm \frac{2d_3\bar{R}}{d_2}$. In other words, the warning lines are the "2-sigma" limits.

Sample Size. The determination of the sample size and the frequency of sampling for $\bar{X}$ and R-chart is a difficult problem. Its complete solution depends not only on the various risks inherent in the sampling process but on the costs of inspection, scrap, and rework. In the first level of sophistication of the design of control chart procedures, the determination is principally based on past practice. Often some efficiency of the test is lost in this approximate design practice but, due to its simplicity and convenience, this type of approximate design is the most popular and widely used in industry.

Second Level of Sophistication

The control chart is merely a convenient device for performing a statistical significance test; the in-control description is the null hypothesis H_o:, while the out-of-control description is the alternate hypothesis H_a:. In Level One control chart design, the alternate hypothesis is usually not stated and it is implicitly assumed. The explicit statement of the alternate hypothesis in the Level Two control chart design, however, permits the design of a more precise control chart procedure than might be possible if it were not stated.

The sample size and the frequency of sampling are determined by logically considering the most relevant process characteristics. During the analysis and design of the sampling procedure, cost parameters are not explicitly taken into consideration.

Illustration. Consider a process which is subject to tool wear. The product is produced in discrete units and has an upper and a lower specification limits, USL and LSL, based on a measurement type quality characteristic, X. Thus, X may represent the diameter of a cylindrical piece being turned on a lathe. The distribution of the random variable X can be satisfactorily described by some density function, f(X), which is assumed to be normal. The null and alternate hypotheses can be stated in the following manner:

Process in-control (H_o:)

$$\text{Process mean} = \mu + \gamma t$$

$$\text{Process variance} = \sigma^2$$

and, γ = a positive constant giving the shift in the process mean per unit time due to tool wear.

Process out-of-control (H_a:)

$$\text{Process mean} = \mu + \gamma t + k\sigma$$

$$\text{Process variance} = \sigma^2$$

On the basis of the above-mentioned process characteristics, the following semi-quantitative design improvements over a Level One design can be made.

i) Whenever any change in the setting of the process mean is to be effected, the process mean should be centered at (LSL + 3σ) in order to maximize the time between the adjustment of the process mean and the occurrence of defective units.

ii) If the maximum of 100p per cent defective items can be tolerated, let k' be the value of Z in the normal table such that it is exceeded in the positive direction by 100p per cent of the cases. To insure that a shift of the process mean to USL - $k'\sigma$ or LSL + $k'\sigma$ is detected with a probability of 0.95, the control limits should be set at USL - $(k' + \frac{1.645}{\sqrt{n}})\sigma$ and LSL + $(k' + \frac{1.645}{\sqrt{n}})\sigma$. There will be little danger of producing excessive defective items as long as the sample mean lies between the control limits.

The average life of the tool can be established and tools may be replaced at the end of every such period, but the control charts are retained to afford protection against unusual behavior of the process mean.

From the illustration given above, it is obvious that the necessary requirements for the design of quality control procedures at the Second Level of sophistication are:

1) the explicit statement of the null and alternate hypotheses;

2) analytical consideration of the most relevant process characteristics; and

3) the prudent and logical use of subjective judgment.

<u>Third Level of Sophistication</u>

The design of a control chart procedure from the Level Three standpoint attempts to consider all the important factors which affect the process both economically and physically. Thus, full consideration is given to the important economic facets of the process in question.

The major difficulty in the Level Three control chart design procedure lies in the fact that it is difficult to formulate a model which represents the behavior of the process with sufficient accuracy, and which can be manipulated easily with the existing mathematical techniques. The formulation of the model requires an analyst with a high degree of mathematical ability, logical reasoning power, and an intuitive judgment.

<u>Illustration</u>. Consider an illustrative example in the economic design of $\bar{X}$ charts used to maintain current control of a process. A product is produced in discrete units and has an upper and a lower specification limits, USL and LSL, based on a measurement type quality characteristic X. The null and alternate hypotheses can be stated in the following way.

Process in control (H_o:)

$$\text{Process mean} = \mu$$

$$\text{Process variance} = \sigma^2$$

Distribution of the random variable X is approximately normal.

Process out-of-control (H_a:)

$$\text{Process mean} = \mu + \gamma t$$

$$\text{Process variance} = \sigma^2$$

Distribution of the random variable X is approximately normal.

The various process characteristics are summarized as follows:

h = running hours between samples.

n = sample size.

ρ = production rate in units per running hour.

k' = the number of standard errors of the mean between μ and a control limit, or

$$= \frac{UCL - \mu}{\sigma / \sqrt{n}} = \frac{\mu - LCL}{\sigma / \sqrt{n}}$$

assuming that the upper and lower control limits are equidistant from μ.

λ = "arrival rate" of assignable causes of variation.

α = probability of making a type I error when the process is in control.

ν = the proportion of the time the process is in-control.

When the process goes out of control:

β_i = the probability of failing to detect the shift on the i'th sample after the inception of an assignable cause.

Z_i = the number of standard errors of the mean by which the process mean has shifted from μ at the time of taking the i'th sample.

η_i = average number of defective units produced between (i - 1) and i'th sample after the process has gone out of control.

h_1 = the expected operating time of the process between the inception of an assignable cause and the first control sample.

θ = the expected elapsed time between the inception of an assignable cause and its detection.

γ = a constant, giving the shift in the process mean per running hour when the process is running out of control.

The relevant cost parameters are summarized as follows:

a + bn = sampling costs in dollars for a sample of size n. It is assumed that a part of the sampling cost is fixed for each sample, such as the cost of sampling and computation. The cost of inspection and part of the cost of sampling and computation vary with the size of sample.

c = loss in dollars for scrap and rework for each defective unit produced when the process is out of control.

ϕ = an assumed factor proportionate to the cost of search for an assignable cause = $\left(\frac{4 - Z_i}{4}\right)^2$, where $(0 \leq \phi \leq 1)$.

V = a factor necessary to convert ϕ from a relative number to a dollar value.

C_s = sampling cost per running hour = (a + bn)/h.

C_o = operating cost per running hour. This is the cost of looking for trouble.

C_d = cost of defective materials per running hour.

C = total relevant process quality costs per running hour. Cost factors, which are unaffected by the choice of n, h and control limits, are not considered.

ψ = expected total cost of looking for trouble when the process goes out of control.

Assumptions. It will be assumed that in the operation of an $\bar{X}$ chart the rule will be followed of taking action only when a sample point falls outside the control limits. The operating cost is the cost of looking for trouble when a point falls outside the control limits. If the shift in the process mean is very small, the probability of finding the trouble is very small and hence the cost of looking for it is large. On the other hand, if the shift in the process mean is very great, the source of trouble will ordinarily be obvious. The total cost of looking for trouble is assumed to vary with the constant V, and the positive quantity ϕ, which is a function of Z_1.

It is also assumed that we have knowledge of the risk of occurrence of an assignable cause. The probability of its non-occurrence before time t, when starting from a state of control, is $e^{-\lambda t}$ and the probability of its occurrence in the interval t to t $+\Delta t$ is approximately $\lambda e^{-\lambda t}\Delta t$. The average time before an assignable cause occurs is $1/\lambda$.

Given the occurrence of an assignable cause between m and (m + 1) sample, Duncan (12) has shown that the expected time of occurrence of the assignable cause is approximately $(\frac{h}{2} - \frac{\lambda h^2}{12})$ hours after taking the m'th sample. Or,

$$h_1 \doteq h - (\frac{h}{2} - \frac{\lambda h^2}{12}) = \frac{h}{2} + \frac{\lambda h^2}{12}$$

Let the control chart be maintained to detect a single assignable cause that occurs at random and results in a change in the process mean of known magnitude. Also, the process begins in a state of control both at the outset and at any time an assignable cause is found and rectified. The rate of production is assumed to be sufficiently high so that when the process is running out of control, the drift in the process mean is negligible among the consecutive items comprising a sample.

To facilitate the analysis of this problem, the assumption will be made that whenever an assignable cause occurs, it occurs h_1 hours before the next control sample. In addition, the magnitude of γ, the rate of shift of the process mean, is large enough to render negligible the probability of not detecting the out-of-control state of the process by the second control sample after the inception of an assignable cause.

Objective. It will be attempted to formulate a theoretical basis for determining three important parameters for the optimal design of $\bar{X}$ charts — the sample size, the frequency of sampling and the distance of the control limits from the target process mean. The design will endeavor to minimize the long run average net loss per unit of time, which will be taken as the definition of optimality. A mathematical function for measuring the average net loss from the process will be formulated.

Procedure. When the process is in control

$$\alpha = 2\int_{k'}^{\infty}(1/\sqrt{2\pi})\exp(\, - \, t^2/2)\,dt$$

Given that an assignable cause has occurred, the probability of not detecting the shift on the first control sample after the inception of the assignable cause,

$$\beta_1 = 1-\int_{-\infty}^{-k'-Z_1}(1/\sqrt{2\pi})\exp(-t^2/2)\,dt + \int_{k'-Z_1}^{\infty}(1/\sqrt{2\pi})\exp(-t^2/2)\,dt$$

$$\doteq \int_{k'-Z_1}^{\infty}(1/\sqrt{2\pi})\exp(-t^2/2)\,dt,$$

since the first term will be practically negligible. In this expression, $Z_1 = h_1\gamma\sqrt{n}/\sigma$.

If, however, the shift in the process mean is not detected in the first control sample after the assignable cause has occurred, the probability of failing to detect the shift in the second sample is given by

$$\beta_2 \doteq 1-\int_{k'-Z_2}^{\infty}(1/\sqrt{2\pi})\exp(-t^2/2)\,dt,$$

where $Z_2 = (h + h_1)\gamma\sqrt{n}/\sigma$.

After the occurrence of an assignable cause, the probability that it will be detected in the first and second sample are $(1-\beta_1)$ and $\beta_1(1-\beta_2)$, respectively. The average time the process will be out of control before a sample point falls outside the control limits is

$$\theta = h_1 + h(\beta_1 + \beta_1 \cdot \beta_2 + \ldots)$$

In many repetitions, the proportion of time the process will be in control is

$$\nu \doteq \frac{1/\lambda}{1/\lambda + \theta}$$

assuming a negligible downtime to rectify the assignable cause. Every time the process goes out of control, the expected total cost of looking for trouble is

$$\psi = V[(1-\beta_1)\phi_1 + \beta_1(1-\beta_2)\phi_2 + \ldots].$$

The expected number of false alarms per hour of operation will be approximately $\alpha\nu/h$. Hence, the average cost of looking for trouble per running hour, when no trouble exists, is

$$V\phi_0\alpha\nu/h = V\alpha\nu/h$$

The total operating cost per running hour is given by

$$C_o = \frac{1}{h}\,[V a \nu + \psi(1-\nu)] \doteq \frac{V}{h}[a\nu + (1-\nu)\left\{(1-\beta_1)\phi_1 + \beta_1(1-\beta_2)\phi_2\right\}] .$$

Consider the number of defective items produced when the process goes out of control.

$$\eta_1 = \sum_{j=1}^{[\rho h_1]} \int_{L_1}^{\infty} (1/\sqrt{2\pi})\exp(-t^2/2)\,dt$$

where $L_1 = \dfrac{USL - (\mu + \gamma j/\rho)}{\sigma}$.

Given that the assignable cause has occurred h_1 hours before the next control sample, the process will produce $[\rho h_1]$ units during the shift of the process mean from μ. The integral inside the summation yields the probability of the j'th unit being defective.

If the shift in the process mean is not detected, the production of defective items continues until the next sample is taken. With probability β_1, therefore, the process is exposed to the production of defective units until $[\rho h]$ units later.

$$\eta_2 = \beta_1 \sum_{j=1}^{[\rho h]} \int_{L_2}^{\infty} (1/\sqrt{2\pi})\exp(-t^2/2)\,dt$$

where $L_2 = \dfrac{USL - [\mu + \gamma(h_1 + j/\rho)]}{\sigma}$.

Considering η_3, η_4, to be negligible, the total loss due to the production of defective items for each occurrence of an assignable cause is given as $c(\eta_1 + \eta_2)$. Therefore,

$$C_d \doteq c(\eta_1 + \eta_2)(1-\nu).$$

The long-term total cost function, for each hour of operation of the process, can be written as

$$C = C_s + C_o + C_d \ ,$$

$$\text{or, } C \doteq \frac{a+bn}{h} + \frac{V}{h}\left[\alpha\nu + (1-\nu)\left\{(1-\beta_1)\phi_1 + \beta_1(1-\beta_2)\phi_2\right\}\right] + (1-\nu)c(\eta_1 + \eta_2)$$

$$\doteq \frac{1}{h}\left[a + bn + \frac{\alpha V}{1+\theta\lambda} + \frac{V\ \theta\lambda}{1+\theta\lambda}\left\{(1-\beta_1)\phi_1 + \beta_1(1-\beta_2)\phi_2\right\}\right] + \frac{\theta\lambda c}{1+\theta\lambda}(\eta_1 + \eta_2)$$

By methods of mathematical analysis, reasonable approximations of the "optimal" values of n, h, and the control limits can be obtained for specified values of a, b, c, V, ρ, LSL and USL, γ, μ, λ, and σ. It will be almost impossible to obtain the absolute optimal values. The range of the values of independent variables will have to be confined to realistic values and approximations in computations should be used to provide relatively simple procedures for the solution of the problem. The basic motivation in evolving the Level Three design approach is that if an economic process model and a process quality model can be formulated and combined to reasonably represent the real-world situation, the intuitive decisions regarding the selection of the sample size, the sample frequency and the control limits can be replaced by more logical and efficient decisions based on the relevant cost and process parameters. The development of the Level Three design of control chart procedures has been extremely limited up to date. This limited development has probably been due to the rapidly increasing complexity of the models as the three levels of design are progressively considered.

Process Quality Models

A comprehensive set of six process quality models is suggested below in order to:

1) assist in the formulation of a specific model for a process quality problem,

2) serve as a means of classifying the existing literature on the optimizing techniques of quality control procedure, and

3) point out the areas where additional research may profitably be directed.

The following models are expressed in terms of quality characteristics which are measurable on a continuous scale, but they can be easily extended to attributes data as well. In order to avoid duplication of the models and to minimize confusion in interpreting the models, quality characteristics existing only on a continuous scale are considered.

Model I

$$X(t) = \mu(t) + \phi_t k\sigma + \epsilon_t ,$$

where

$X(t)$ = an estimate of the process mean, $\mu(t) + \phi_t k\sigma$, based on n observations at time t.

$\mu(t)$ = an estimated or a known function of time according to which the process mean varies. The drift can be caused by tool wear, depletion of a chemical, etc. The most important case is when the process mean does not change, i.e., $\mu(t) = \mu$.

ϕ_t = 0, when the process is in control.

ϕ_t = $\pm$ 1, when the process is out of control.

$k > 0$.

ϵ_t = random error inherent in the process and the measurement system of the process. It is statistically independent of t and assumed to be normally distributed with mean zero and variance σ^2/n.

Model I thus represents a process which characteristically goes out of control by a fixed shift in the process mean of magnitude $k\sigma$ units, from $\mu(t)$ to $\mu(t) \pm k\sigma$. This model may appear to be unrealistic, but, under some circumstances, it is extremely valuable. If no reasonably true approximation can be made of the exact nature of the shift in the process mean when the process goes out of control, this model will be used. Moreover, if it is not important to detect small shifts in the process mean while a shift of magnitude greater than $k\sigma$ will lead to the production of defective items of proportion greater than the acceptable quality level, Model I will describe the necessary conditions of the quality control procedure in a useful way. Due to the inherent simplicity and convenience of use, this model has been most widely studied.

Model II

$$X(t) = \mu(t) + \phi_t f(t-t^*) + \epsilon_t$$

where $X(t)$, $\mu(t)$, and ϵ_t are the same as in Model I, and

ϕ_t = 0, when the process is in control.

ϕ_t = 1, when the process is out of control.

$f(t)$ = a known or estimated non-zero function of time.

t^* = the time at which the process goes out of control.

Model II differs from Model I in the nature of the behavior of the process mean when the process goes out of control. In Model I, when the process goes out of control, the process mean jumps from $\mu(t)$ to $\mu(t) \pm k\sigma$, denoting an instantaneous shift in the process mean. In Model II, with the inception of an assignable cause of variation, the process mean has an additional variation which is a known function of time commencing with the initiation of the assignable cause. Typical examples of the function, $f(t)$, will be linear or higher polynomial or even trigonometric.

If, when the process goes out of control, the approximate nature of the shift in the process mean can be determined, Model II should be used as it affords a more precise quality control procedure than Model I. In the absence of an estimate of the nature of a shift, however, Model I will have more validity.

<u>Model III</u>

$$X(t) = \mu(t) + \omega_t + \epsilon_t$$

where $X(t)$, $\mu(t)$ and ϵ_t are the same as in Models I and II, and

ω_t = a random variable assumed to be normally distributed with

$E(\omega_t) = 0$, and

$E(\omega_t \; \omega_{t-m}) = \rho_m \theta_i^2 \sigma^2$,

where the constants, ρ_m, are independent of t.

$\rho_m = 1$ when $m = 0$.

$0 \leq \rho_m < 1$ when $m > 0$.

$\theta_i = \theta_1 \geq 0$ when the process is in control.

$\theta_i = \theta_2 > \theta_1$ when the process is out of control.

Model III describes a covariance stationary Normal process as defined by Parzen (41). It differs from models I and II in the basic assumption about the shifts in the process mean. In the first two models, the shifts were assumed to be either fixed or a determinable function of time and as such, these two models are essentially deterministic. In Model III, however, the magnitude of a shift in the process mean is a time dependent random variate. This is a very realistic model since it allows for assignable causes of all possible magnitude on a stochastic basis. It also permits the in-control process mean to be shifting in a random manner.

By a suitable selection of parameters, Model III can be modified to represent different types of processes. For example, if $\rho_m = 0$ for all $m > 0$, the model describes a time independent process in which successive observations made on the process can be regarded as successive independent random variables. If $\rho_m \neq 0$ for some $m > 0$, the successive observations of the process are assumed to be a run of dependent random variables. If θ_1 is zero, it is implied that when the process is in a state of control, the process mean behaves according to the function $\mu(t)$, except for the random variation ϵ_t, and shifts from $\mu(t)$ by a non-zero random amount ω_t when the process goes out of control. On the other hand, if $\theta_1 > 0$, the process is assumed to be always out of control in the usual sense and the purpose of the control chart is to detect any sustained variation in the process mean which is greater than a predetermined critical value.

Although Model III is one of the most important and realistic models to depict industrial processes in general, the practical difficulties in studying the model are substantial. If the validity of the model can be established for a process and the parameters of the model can be estimated, Model III will yield an efficient process quality control procedure.

Model IV

$$X(t) = \mu(t) + \epsilon_t$$

where X and $\mu(t)$ are the same as in Models I, II, and III, and

ϵ_t = short term random variations inherent in the production process and the measurement system. It is statistically independent of t and is assumed to be normally distributed with zero mean and variance $\phi^2 \sigma^2/n$.

$$1 \leq \phi \leq 1 + \delta \quad \text{when the process is in control.}$$

$$1 + \delta < \phi \quad \text{when the process is out of control, and}$$

$$\delta \geq 0 \ .$$

Model IV principally deals with those processes in which the random variation from item to item is of primary importance. When the process is in-control, the standard deviation of the process is between σ and $(1 + \delta)\sigma$. When the process goes out of control, ϕ is greater than $1 + \delta$, and hence the standard deviation of the process is greater than $(1 + \delta)\sigma$. The reason for explicitly stating δ is to set an upper bound on the process standard deviation. No action need be taken if the variability of the process is less than this bound.

Model V

A group of composite models can be formed by combining Model IV with Model I, II, or III.

In the first three models, it is presumed that only the process mean should be controlled without any regard to controlling the variability of the process. In Model IV, the control of only the process variability is deemed important. Usually, however, the control of both the process variability and the process mean is necessary and hence this model is important.

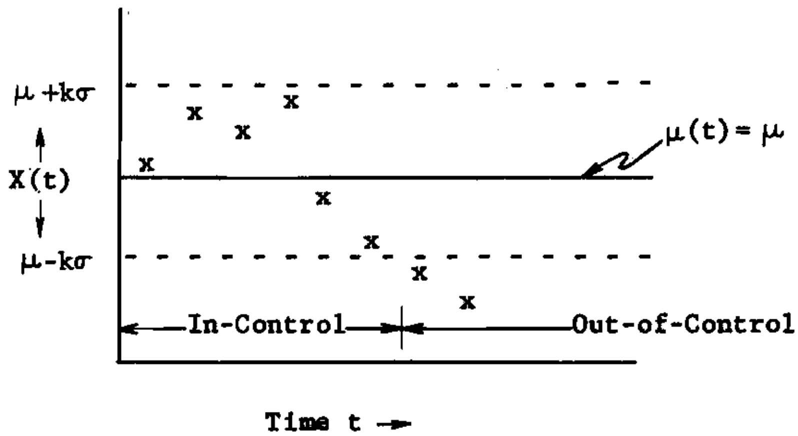
Model VI

Model VI provides for variations in Models I through V in regards to the assumptions about the distribution functions of the random variables involved in these models. In the first five models, the random variables are assumed to be normally distributed.

The assumption of normality, according to the Central Limit Theorem, is usually valid for the random variable ϵ_t, provided the observations comprising a sample are independent and the number of observations in a sample are greater than four or five. In regards to ω_t, however, the normal distribution may well describe the random variate if a comparatively large number of independent causes generate ω_t. If, however, the past data are described better by a different distribution, the assumption of normality should be discarded.

CHAPTER III

MODEL I



$$X(t) = \mu(t) + \phi_t k\sigma + \epsilon_t$$

where X(t) = an estimate of the process mean, $\mu(t) + \phi_t k\sigma$, based on n observations at time t.

$\mu(t)$ = an estimated or known function of time according to which the process mean varies. The most important case, as shown above, is $\mu(t) = \mu$.

ϕ_t = 0, when the process is in control.

ϕ_t = ± 1, when the process is out of control.

k > 0 .

ϵ_t = random error inherent in the process and the measurement system of the process. It is statistically independent of t and assumed to be normally distributed with zero mean and variance σ^2/n.

Model I represents a process which goes out of control by a fixed shift in the process mean of magnitude kσ units. This model often will not represent the actual behavior of the process but, for the purpose of quality control analysis, it will be fully adequate for a great variety of processes. It is extremely valuable in two typical situations.

First, Model I can be widely used when kσ is interpreted as a "serious" shift in the process mean rather than the actual physical nature of the shift. In many processes, the product specifications permit some amount of fluctuations in the process quality level. No action need be taken to restore the quality level to its target value unless the shift in the process mean exceeds a critical value, namely, kσ units. Thus, if the shift in the process mean is less than kσ units, detection is relatively unimportant, while, if the shift is greater than or equal to kσ units, the detection probability should be sufficiently large. Secondly, if it is difficult to estimate the actual behavior of the process, the first model may be used. This model may be used for simplicity where the situation does not warrant the consideration of the more elaborate Models II and III. As the ultimate purpose of quality control procedures is to achieve economy of operation of the process, Model I may be ideally suited for such processes.

The first model is the simplest model and hence it is extremely well suited for statistical analysis and practical application. For these reasons, this model has been most widely studied. It is well suited for various metal working operations, including machine shop production, foundry jobs and metal forming. Various other industries, like woodworking and garment making, which are subject to random fluctuations in quality, can utilize this model effectively.

The estimation of the necessary parameters is made with the help of conventional tools. The standard deviation of the process, σ, and the behavior of the process mean, $\mu(t)$, can be estimated from past data. The value of k can then be determined by analyzing the standard deviation of the process and the specification limits.

The studies pertaining to Model I are listed as follows: Aroian and Levine (1), Aroian (2), Armitage (3), Cohen (10), Duncan (12), Ewan and Kemp (14), Ewan (15), Freund (19), Goldsmith and Whitfield (22), Howell (24), Ito (25), Johnson and Leone (26), Kemp (28,29), King (30), Mitten and Shanoh (31), Moore (32), Page (33, 34, 35, 36, 37, 38, 39, 40), Roberts (46), Scheffe (47), Truax (49, 50), Weiler (52, 53, 54), and Wolfowitz (55). In fact, out of a total of 43 articles considered, 33 articles deal with Model I either partially or fully. Only two articles, Ito (25) and Duncan (12), refer to Model I from a Level Three design standpoint. The rest of the studies are useful in making a Level Two control chart design.

Aroian (2), along with Levine (1), considers the choice of the sample size, the frequency of sampling, and the placement of the control limits. When the process goes out of control, three principal cases are investigated: (i) the probability, γ_i, of detecting the shift on the i'th sample after the inception of the assignable cause remains constant for all i, (ii) γ_i varies with i according to some known function, and (iii) γ_i is a random variable. The average number of units produced between unnecessary interruptions of the process due to type I error is kept constant while the expected number of samples to detect a significant shift in the process mean is minimized for different sample sizes and control limits. Cohen (10), on the other hand, deals with the problem of optimal selection of the frequency of sampling for attributes testing only. Using the notation of Aroian and considering only case (i), as above, Cohen has prepared nomographs to express the relationship

$$\beta = (1-\gamma)^m$$

where β is the probability of not detecting the shift in m consecutive samples.

Page (33, 35) and Weiler (52, 53, 54) considered the problem of optimum sample size and the placement of the control limits for the conventional single sample $\bar{X}$ chart, considering also certain run rules as a substitute of, or along with, the conventional control chart. Page defines Average Run Length (ARL) as

$$L = \frac{\text{Sample Size n}}{\text{Probability of Rejecting } H_o\text{:}}$$

If H_o: is true, then $L_o = n/\alpha$ where α is the probability of making a type I error and L_o denotes the average number of items inspected before rejecting the null hypothesis when it is true. When the process goes out of control, or H_a: is true, $L_1 = n/(1-\beta)$ where β is the probability of making a type II error and hence L_1 denotes the average number of items inspected before detecting the shift in the process mean. Thus, ARL is an indirect measure of the amount of defective articles produced when quality is bad, and of the frequency of interference with the production caused by the process inspection scheme when quality is satisfactory. It is reasonable to choose the inspection scheme such that L_1 is minimized for some given large value of L_0, or, alternatively, L_0 is maximized for some given small value of L_1. Page (33) has given tables from which the desired plans can be selected. The sample sizes and the control limit values are given for different values of k to minimize L_1 for a specified value of L_0, or to maximize L_0 for a specified value of L_1.

Page (35) later concludes that from the point of view of ARL, the theoretically best sample size is often much larger than the customary sizes and it may be often impractical to take such large samples. Moreover, a small sample taken more frequently will spot a very serious change in the process mean quickly. Page suggests the use of charts with warning lines. If only the process mean is to be controlled, Page advocates the use of the rule: chose r, λ, n; take samples of size n; take action if any point falls outside the action lines or if any r out of the last λ points fall outside the warning lines. He proves that, for the rule to be satisfactory, either r = 2 for any λ or $r = \lambda$. For each of the two schemes, Page has tabulated the values of ARL for different values of n, B_1, B_2, k and λ, where the action lines are placed at $\mu \pm B_1 \sigma/\sqrt{n}$ and the warning lines at $\mu \pm B_2 \sigma/\sqrt{n}$. The final conclusion seems to be in favor of using moderate sample sizes. For samples of ten or more, the ARL for k>1 is almost identical to the sample size.

Moore (32) suggests the use of only run rules: choose λ, n; take samples of size n and take action only if λ consecutive points fall beyond a control limit. By comparing ARL calculated on the basis of the above rule and the ARL given by Page, he concludes that his scheme is more sensitive to detect small changes in the process mean than the schemes proposed by Page. Page (40), in a later article, points out that the comparison is not valid due to the different characteristics of the two sets of rules. Page modifies his rule to compare the schemes on a valid basis and concludes that inspection schemes based on control charts with both warning and action lines have almost the same sensitivity as the schemes based only on runs for small changes in the process mean, up to a shift of 0.4σ in the process mean. For larger changes, however, the former is shown to be consistently better.

The results of Weiler's studies (52, 53, 54) have produced similar conclusions. He also studied ways of improving the power of small samples in detecting small shifts in the process mean ($k \leq 1$) by the use of special run rules. He has drawn up power curves to show the power of various sample sizes in detecting shifts in the process mean keeping the average number of false alarms generated a fixed percentage of the number of articles tested irrespective of the sample size. From this study, he concludes that large sample sizes have marked superiority in detecting small shifts in the process mean from the point of view of average run length. For instance, a sample size of ten is superior to a sample size of five for all k less than 1.1.

Weiler's run rules can be stated as follows: chose λ and n; take action if λ successive values fall above the upper control limit at $\mu + B_\lambda\sigma/\sqrt{n}$ or below the lower control limit at $\mu - B_\lambda\sigma/\sqrt{n}$. B_λ is determined so that when the process is in control, an average of 500 samples are required to obtain a significant run outside of either control limit. Weiler has determined the values of λ and $B_\lambda/\sqrt{n}$ for various values of n and k for minimizing the average number of items inspected for the detection of a shift in the process mean of $k\sigma$ units. Savings in the average number of items inspected are quite significant for appropriate values of $\lambda > 1$, using $\lambda = 1$ as a basis of comparison. Weiler also points out the possibility of combining decision rules for different values of λ on the same chart, if the complexity of the control chart is not objectionable.

Ito (25) studied Model I from a Level Three design viewpoint. He deals with the problem of evaluating the optimal sample size and control level for an $\bar{X}$ chart by minimizing the maximum mean risk with respect to the magnitude of the process shift. When the process is in control, the sample mean $\bar{X}$ has a normal distribution with mean μ and variance σ^2/n and the cost of control is

$$In + \alpha C$$

where I = inspection cost for an item.

n = sample size.

α = the probability of a type I error of the chart, determined by the coefficient of the control limit, B.

C = cost of an adjusting action.

When the process mean slips by $k\sigma$, the loss becomes

$$In + NDk + \gamma C$$

where N = number of items produced between two consecutive samples.

Dk = the loss per item produced when the process mean shifts by $k\sigma$.

γ = the probability that $\overline{X}$, an estimate of $\mu \pm k\sigma$, is outside the control limits at $\mu \pm B\sigma/\sqrt{n}$.

The process mean is supposed to shift accidentally from μ to $\mu \pm k\sigma$ with a constant probability P for each sample. Adjusting action is taken as soon as a point on the $\overline{X}$ chart falls outside the control limits. Under these assumptions, the expected risk is given by

$$R = In + \frac{P}{P + \gamma}[NDk + C\gamma(1 + \frac{\alpha}{P})].$$

The objective is to seek values of B, n, and k which make R to be

$$\min_{B,n} \max_{k} R$$

Setting the partial derivatives of R with respect to B, n, and k equal to zero, Ito has constructed diagrams showing the relationship between the process parameters and the optimum values of the control chart parameters B, n, and k. The diagrams show that the optimum control level B is almost determinable by C/DN only. For small values of C/DN, relatively narrow control limits, for example 2-sigma limits, are preferable. The optimal sample size, n, is almost independent of the transition probability P, and is primarily controlled by the chosen value of B.

Duncan (12) studied this model from a Level Three design standpoint and his study has contributed immensely to the design of optimal quality control chart procedures. In his model, the process mean is supposed to be liable to jumps occurring in a Poisson process, and the magnitude of each jump is constant. Moreover, he assumes that a jump in the process mean will be detected and the process mean will be restored to its target value before the next jump can occur.

Duncan has formulated a mathematical function for measuring the average net income from a process operating under the surveillance of an $\bar{X}$ chart and has evolved a procedure for determining the design that maximizes for the process the long run average net income per unit of time, on the assumption that the risk of occurrence of an assignable cause and various cost and income parameters are known. He has, applying the conventional decision rules, tabulated the approximately optimal values of the sample size (n), the interval between samples (h), and the placement of the control limits at $\mu \pm B\sigma / \sqrt{n}$ for variations in the assumptions of such costs as those associated with sampling, searching and eliminating causes of out-of-control variations, the loss incurred when the process runs out of control, assumptions regarding the frequency of occurrence of the out-of-control condition and the magnitude of the shift in the process mean. In each case, he has computed the loss cost in dollars for each 100 hours of operation for both the optimum control chart values and the conventional chart values. These loss cost differences exhibit clearly the importance of considering the economic factors in making specific recommendations about the parameters n, h, and B. They also point out that when economic as well as statistical factors are considered, no single optimal recommendation of n, h, and B can be made for a set of statistical characteristics of the process.

Roberts (46) advocates the use of geometrical moving average charts. A geometrical moving average gives the most recent observation the greatest weight and the weights of all the previous observations decrease according to a geometric progression from the most recent observation back to the first. Let Z_j denote the moving average of the observations of the process mean $\bar{X}_i$ (for all $i \leq j$) at time j, and let $Z_o = \mu$. If the most recent observation is assigned a weight r,

$$Z_j = (1-r)^j\mu + r(1-r)^{j-1}\bar{X}_1 + r(1-r)^{j-2}\bar{X}_2 + \dots + r(1-r)\bar{X}_{j-1} + r\bar{X}_j$$

$$= r\bar{X}j + (1-r)Z_{j-1} .$$

A graphical procedure for generating the moving averages directly on the control chart is described. Roberts gives the curves of the average run lengths for various values of k and different test procedures. Although he concludes that tests based on geometric moving averages compare most favorably with the conventional tests, the increase in the power of the test, as shown in his diagrams, is hardly sufficient to warrant the increase in complexity caused by the use of moving average charts.

One of the most important recent developments in statistical quality control procedure has been the use of cumulative sum schemes. Page (36,37) develops the scheme from a test for detecting a change in a parameter occurring at an unknown point. Consider a sample of independent observations in the order in which they are obtained, $x_1, x_2, \ldots, x_n$. It is sometimes required to test the null hypothesis that all the observations come from the same population with distribution function $F(x|\theta)$, against the alternative hypothesis that $x_1, x_2, \ldots, x_m$ $(m<n)$ come from the population described by $F(x|\theta)$ and $x_{m+1}, x_{m+2}, \ldots, x_n$ from $F(x|\theta')$, $(\theta' \neq \theta)$. If m is known, there is no problem in testing the hypothesis. If m is unknown, Page proposes the cumulative sum test as a test of hypothesis. Assume that θ is initially known. Use the test statistic

$$h = \max_{o \leq r \leq n} [S_r - \min_{o \leq i \leq r} S_i] \qquad \text{where } S_r = \sum_{j=1}^{r} (x_j - \theta) \text{ and } S_o = 0.$$

Reject the null hypothesis if h is significantly large.

On the basis of the cumulative sum test, Page (34) develops an inspection scheme. A one-sided process inspection scheme is first stated and then Page shows that the rule is equivalent to a sequence of Wald (51) sequential tests. The principle is then extended to two-sided tests by considering the simultaneous application of two Wald sequential tests.

Johnson and Leone (26) give much of the mathematical development behind cumulative sum charts. To consider control charts for the process mean, let the observed values of the random variables x be normally distributed with mean μ and standard deviation σ. If H_o: specifies $\mu = \mu_o$ and H_a: specifies $\mu = \mu_o + k\sigma$ $(k > 0)$, then for a random sample of m values $\bar{x}_1$ $\bar{x}_2, \ldots, \bar{x}_m$, the likelihood ratio (l_{am}/l_{om}) of probabilities of observing $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$, when H_a: or H_o: is true, respectively, is given by

$$\frac{l_{am}}{l_{om}} = \frac{(\sqrt{2\pi}\sigma)^{-m} \exp\left[-\frac{1}{2\sigma_{\bar{x}}^2}\sum_{i=1}^{m}(\bar{x}_i - \mu_o - k\sigma_{\bar{x}})^2\right]}{(\sqrt{2\pi}\sigma)^{-m} \exp\left[-\frac{1}{2\sigma_{\bar{x}}^2}\sum_{i=1}^{m}(\bar{x}_i - \mu_o)^2\right]}$$

$$= \exp\left\{\frac{1}{2\sigma_{\bar{x}}^2}\left[2k\sigma_{\bar{x}}\sum_{i=1}^{m}(\bar{x}_i - \mu_o) - mk^2\sigma_{\bar{x}}^2\right]\right\}$$

from which

i) if $\frac{1}{\sigma_{\bar{x}}}\sum_{i=1}^{m}(\bar{x}_i - \mu_o) \leq \frac{1}{k}\ln\frac{\beta}{1-\alpha} + \frac{1}{2}mk$ accept $\mu = \mu_o$.

ii) if $\frac{1}{\sigma_{\bar{x}}}\sum_{i=1}^{m}(x_i - \mu_o) \geq \frac{1}{k}\ln\frac{1-\beta}{\alpha} + \frac{1}{2}mk$ accept $\mu = \mu_o + k\sigma_{\bar{x}}$,

where α and β are the probabilities of the errors of first and second kind, respectively. Hence, if the points

$$\left[\ m, \frac{1}{\sigma_{\bar{x}}}\sum_{i=1}^{m}(\bar{x}_i - \mu_o)\right]$$

are plotted on a graph paper, then the "continuation region" lies between two parallel straight lines, each inclined at an angle $\tan^{-1}(\frac{1}{2}k)$ to the axis of m and with intercepts $\frac{1}{k}\ln\frac{\beta}{1-\alpha}$ and $\frac{1}{k}\ln\frac{1-\beta}{\alpha}$, respectively, on the axis of $\frac{1}{\sigma_{\bar{x}}}\sum_{i=1}^{m}(\bar{x}_i - \mu_o)$. In the cumulative sum charts, however, only the second straight line is retained as the objective of the test is to differentiate between the rejection region and the continuation region.

This is basically a one-sided test, and if a two-sided test is desired, Armitage (3) proposed the simultaneous application of two Sequential Probability Ratio Tests, one discriminating between H_o: and H_a: for $\mu = \mu_o + k\sigma$ and the other discriminating between H_o: and H_a: for $\mu = \mu_o - k\sigma$. On the control chart, the test of hypothesis is applied graphically with the help of a V-mask. The V-mask, with half-angle θ, is placed on the chart with its axis parallel to the axis of m and the apex at a lead distance d units from the last point plotted. If all the previous plotted points are within the 'V' of the mask, the null hypothesis is accepted and if any plotted point falls outside the limbs of the 'V', the null hypothesis is rejected. The important parameters of the mask, d and θ, can be obtained by a "cut-and-try" process or by analytical methods. According to the cut-and-try method, considerable past data in the cumulative sum form is reviewed. A mask is developed such that during periods in which satisfactory product was produced, the mask will not signal action, or, all points remain within the 'V'. Then, during periods when finished product was unsatisfactory, the mask should be fashioned so as to signal action as quickly as possible. Another procedure, which has very doubtful validity, is to compare both the standard chart and the cumulative sum chart, or the cusum chart, for the same data. If any point on the standard control chart falls on or very near any control limit, the same plotted point on the cusum chart should also fall on or near the corresponding limb of the V-mask. With the help of such points, the V-mask can be constructed. According to the analytical method of construction of the V-mask,

$$\theta = \tan^{-1} \tfrac{1}{2}k, \text{ and}$$
$$d = \frac{2}{k^2} \ln \frac{1-\beta}{\alpha/2} \doteq - \frac{2}{k^2} \ln \frac{\alpha}{2} \qquad (\text{if } \beta \text{ is very small}).$$

In the case of one-sided schemes, it has been shown by Page (34, 38) and Kemp (28) that the distribution of the run length is approximately geometric when the ARL is large. Page showed that the distribution function of the run length, L, is approximated for large m by

$$\Pr (L \leq m) = 1 - P^{v+1}$$

where $v = m/N_1$, P is the probability of accepting H_o:, and N_1 is the average sample number conditional upon the acceptance of H_o: using the Wald test. If

$$L = N/ (1-P)$$

where N is the unconditional average sample number and if $N_1 \doteq N$, Kemp's result is obtained:

$$\Pr (L \leq m) = 1 - \exp (-m/L)$$

Ewan and Kemp (14) have shown that instead of plotting cumulative sum of deviations from the target value μ, $\sum(\bar{x}_i - \mu)$, it is preferable to plot $\sum(\bar{x}_i - \mu_1)$, where μ_1 is a reference value. If μ_1 is taken to be a convenient quantity near $(\mu + \frac{k\sigma}{2})$, the ARL, when the process is centered at μ, is near its maximum for a given value of ARL when the process is centered at $\mu + k\sigma$.

The average run length of the two-sided scheme is derived by Kemp (28) wherein he shows that only one decision boundary can be crossed at a time by the cusum chart. The average run length is given by

$$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2}$$

where L_1, L_2, and L are the average run lengths before reaching a rejection decision with regard to the upper, lower and either limb of the V-mask respectively. From an analysis of the run length distribution, Ewan and Kemp (14) conclude that the distribution has the desirable property of having a high probability of obtaining a run length near the average level while the probability is low for obtaining a run length which is considerably larger than the average level. From the nomograms of ARL, it is also found that the relationship between the sampling interval and the sample size, for fixed ARL, is very nearly linear. It is, therefore, preferable to sample frequently with a small sample size in order to guard against large shifts in the process mean.

Freund (19), Johnson and Leone (26), Goldsmith and Whitfield (22), and Page (34,38) have determined the average run lengths of cusum charts for different values of k, n, and the control limits. The conclusions derived by the different authors are almost identical. Cumulative sum schemes are considerably more sensitive in detecting small shifts in the process mean than corresponding conventional $\overline{X}$ charts. For shifts between 0.5σ and 2.0σ, cusum charts give more rapid indication of the change, while, for larger deviations, standard charts have lower ARL. When deviations in the process mean exceed 2.5σ, however, the ARL is very small in any case and the choice of the scheme in this region would depend probably on considerations other than the small absolute differences in ARL. According to Johnson and Leone, the cusum charts are more sensitive if the allowable probability of a type I error, α, is small. The advantage decreases sharply as α increases.

The most fundamental advantage of a cusum chart is the ease with which the changes in the process mean, as well as the point of initiation of the change, can be detected visually by a change in the slope of the chart. The precision of this estimation procedure is not well established but it seems to work in practice. Truax (49,50) suggests that the shift in the process mean can be estimated from the slope of the plotted points.

Ewan (15) and Truax (49,50) consider the application of cumulative sum charts on an extremely practical basis without considering the principles in any detail. The recommendations suggested by them are similar. The principal disadvantages of cusum charts are the increased complexity of application and the difficulties of choosing the proper scale factor for plotting the points on the control chart. Ewan suggests that a standard control chart should be used if inspection is inexpensive, extreme simplicity in the use of the control charts is required, and the magnitude of a "serious" shift in the process mean is greater than 2σ. If tests are relatively expensive, and the type of change to be detected is a fairly sudden shift in the mean level of the parameter while the change is sustained for an average of at least approximately 5 to 10 sampling intervals, then a cumulative sum chart should be used.

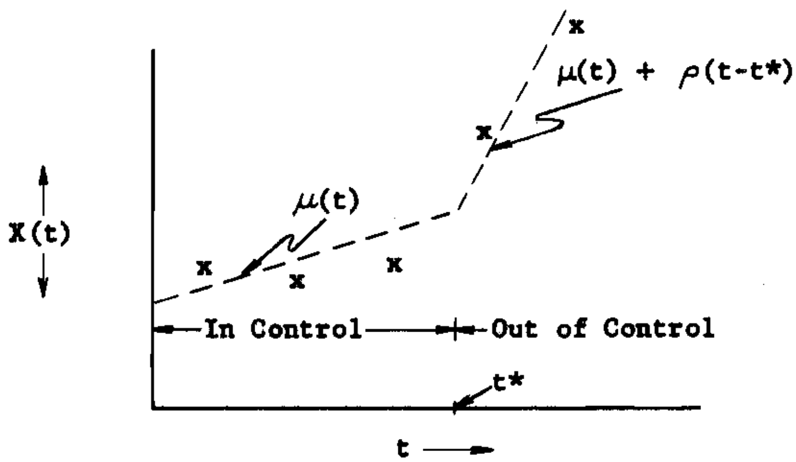
Some practical suggestions, based on the experience of Ewan and Truax, are also given for the construction and use of cusum charts. The scale factor should be so chosen that a change of 2σ in the process mean should, for the plotted points, result in a slope of approximately 45° on the control chart. This compression of the vertical scale is necessary if the points are to be conveniently plotted and to reduce system noise without sacrificing the detection power.

Johnson and Leone (26) have extended the principle of cumulative sum schemes to control Binomial and Poisson variables, or, 'p' charts and 'c' charts. A comprehensive set of formulas and tables is given for constructing different cumulative sum schemes with appropriate scaling factors. Page (39) has also considered the use of cusum charts to control processes where data are obtained from a go-no-go gage situation. He points out that the high sensitivity of the cusum schemes for small departures of the process mean is a drawback if the process capability is much better than that required by the product specifications.

The brief discussion of the literature indicates that a wealth of information is available for a control chart design where process quality Model I is appropriate and where a "serious" value of k can be stipulated. Much of the literature pertaining to Model I refers to the design of a control chart procedure from a Level Two standpoint. Due to the interaction of statistical and economic factors, great care must be exercised in applying recommendations given in the literature that are based on statistical considerations alone.

CHAPTER IV

MODEL II



$$X(t) = \mu(t) + \phi_t f(t-t^*) + \epsilon_t$$

where

X(t) = an estimate of the process mean based on n observations at time t.

$\mu(t)$ = an estimated or known function of time according to which the process mean varies.

ϕ_t = 0, when the process is in control.

ϕ_t = 1, when the process is out of control.

f(t) = a known or estimated non-zero function of time.

t* = the time at which the process goes out of control.

ϵ_t = random variation inherent in the process and the measurement system. It is statistically independent of t and assumed to be normally distributed with zero mean and variance σ^2/n.

Model II is identical to Model I except only in the assumption about the nature of the behavior of the process mean when the process is out of control. In Model I, when the process goes out of control, the process mean jumps from $\mu(t)$ to $\mu(t) \pm k\sigma$, so that there is an instantaneous shift in the process mean. In Model II when the process goes out of control, the function, X(t), has an additional component which is a known function of time commencing with the inception of the assignable cause. Typical examples of the function f(t) will be linear or higher polynomial or even trigonometric.

If the nature of the shift in the process mean can be estimated, Model II should be used as it affords a more sensitive quality control procedure than Model I. If, however, it is difficult to estimate the function f(t), or if the costs of estimation of f(t) are greater than the benefits to be derived from the increased sensitivity, Model I will be more appropriate.

Model II can be applied in controlling the quality of various chemical processes. Consider, for example, a catalytic process in which the normal rate of change of some characteristic of the output due to the aging of the catalyst may be accelerated with poisoning of the catalyst at some instant of time. Assuming that the rate of change of the characteristic X can be represented by a linear function of time, $\mu + \gamma t$ may represent the function $\mu(t)$ when the process is in control. With the poisoning of the catalyst, $\phi_t f(t-t^*)$ may be represented by $\rho(t-t^*)$, where t* is the instant of the sudden poisoning of the catalyst. Thus, when the process goes out of control, X(t) may be represented by $\mu + \gamma t + \rho(t-t^*) + \epsilon_t$.

Model II can similarly describe the metal cutting rate of a machine tool where γ and ρ are two constants depending on the condition of the coolant, the type of tool being used, etc. Many aspects of papermaking industry, steel industry, textile manufacture, metallurgical processes and other similar industries may be represented by Model II. At this stage, it should be understood that both Models I and II are basically deterministic in nature. In Model II, it is assumed that the behavior of the process mean can be predicted as a determinable function of time in both out-of-control and in-control situations. The random variation inherent in the process is ϵ_t which remains the same both before and after the occurrence of an assignable cause.

The parameters of the process can be estimated with the help of regression analysis of the past production data. The function $\mu(t)$ should be assumed to be constant and equal to μ unless γ or any other coefficient of a higher order of t is significant. When the process is running out of control, the slope of X(t) can be estimated, from which the parameter ρ can be determined. Data should be periodically analyzed, even after the establishment of the model, to confirm or alter the values of the parameters.

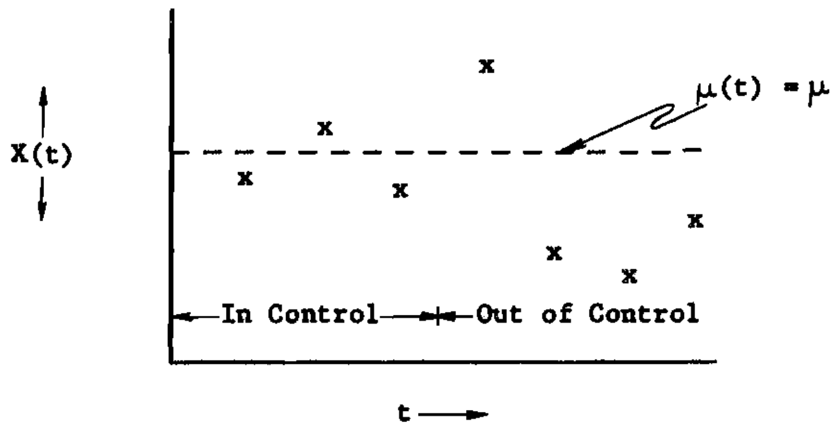
The control procedure based on Model II will usually utilize control limits placed at a distance of $B\sigma / \sqrt{n}$ from the central line, which may be parallel to or at an angle to the time scale. The parameter B will be proportional to ρh, where h is the sampling interval and the factor of proportionality will be determined by considering the relevant process and economic characteristics of the process. One of the advantages of Model II control chart design is that when the process has been detected to have gone out of control, the time of occurrence of the assignable cause can be quite accurately determined and the finished products produced subsequently may be subjected to 100 per cent inspection.

Aroian (2), in his study of the optimal selection of the sample size and the placement of the control limits for a fraction defective chart, has briefly mentioned this model. The function, f(t), has been illustrated as a linear or an oscillatory function. No conclusion, however, has been drawn from the study.

This model has received little attention from researchers and control chart users in the past. The writer feels that Model II warrants much greater attention, particularly in those industries where the process behavior can be represented as a function of time.

CHAPTER V

MODEL III



$$X(t) = \mu(t) + \omega_t + \epsilon_t$$

where

$X(t)$ = an estimate of the process mean based on n observations at time t.

$\mu(t)$ = an estimated or known function of time according to which the process mean behaves.

ω_t = a random variable assumed to be normally distributed with

$E(\omega_t) = 0$.

$E(\omega_t\, \omega_{t-m}) = \rho_m \theta_i^2 \sigma^2$ where the constants, ρ_m, are independent of t, and

$\rho_m = 1$ when $m = 0$.

$0 \leq \rho_m < 1$ when $m > 0$.

$\theta_i = \theta_1$ ≥ 0 when the process is in control.

$\theta_i = \theta_2$ $> \theta_1$ when the process is out of control.

ϵ_t = random variation inherent in the process and the measurement system. ϵ_t is statistically independent of t and normally distributed with zero mean and variance σ^2/n.

Almost all literature, dealing with the optimization techniques of quality control procedure, has been centered around the conception of the state of "statistical control" as originally defined by Shewhart (48). In such a system, the successive observations made on an industrial process can be regarded as successive independent samples from a statistical population. Now that the theory of stochastic processes has grown into a well-rounded body of theory, it is appropriate to consider industrial processes, whose natural intrinsic variability is best described as a run of dependent random variables. The ever more extensive application of feedback control mechanisms, often embodying time delay, implies that such processes are growing more and more common. It is now necessary to use the control chart as itself an element of a feedback loop, which not only signals the departure of the process from its target value, but also indicates the magnitude of the departure, and what amount of correction needs to be applied to the process to bring it back to a state of control.

Model III attempts to incorporate some of these features and thereby break loose from the conventional quality control procedures. It describes a covariance stationary Normal process as defined by Parzen (41). The third model differs from the first two models in the basic assumption about the shifts in the process mean when the process goes out of control. While, in the first two models, the shifts were assumed to be either fixed or a determinable function of time, in Model III, the magnitude of the shift is a time dependent random variate. This is a very realistic model since it allows for assignable causes of all possible magnitudes on a stochastic basis and also permits the in-control process mean to be shifting in a random manner.

By a suitable selection of the parameters of the model, it is possible to modify the model to represent different types of processes. For example, if it is assumed that $\rho_m = 0$ for all $m > 0$, the model reduces to the description of a time independent process in which there is no autocorrelation between successive observations. If $\rho_m > 0$ for some $m > 0$, the successive observations of the process are assumed to be a run of dependent random variables.

The parameter θ_1 determines the behavior of the process mean when the process is in a state of control. If θ_1 is zero, it is implied that the process behaves according to the function $\mu(t)$ with the only inherent randomness being expressed by ϵ_t. On the other hand, if θ_1 is greater than zero, the model describes a process which exhibits an erratic variation in the process mean in addition to the random error ϵ_t and hence the process is always out of control in the usual sense. The parameter θ_2 represents a predetermined critical value of the variable shift in the process mean which it is important to detect.

Although Model III is one of the most important and realistic models, especially to depict process industries, the practical difficulties in studying and validating the model are substantial. The most difficult aspect in the study of this model lies in the estimation of the parameters of the model. If, however, the validity of the model can be established for a process and the relevant parameters can be estimated, Model III will yield an efficient quality control procedure. This model is quite appropriate in the paper, textile, chemical, and electronics industries, where fluctuations in the raw materials and manufacturing operations are both frequent and significant.

As an illustrative example, the manufacture of paper is considered. Paper manufacture is essentially a chemical process in which the basic raw material is processed in batches. The control procedure consists of (1) analysis of the raw material, (2) analysis of samples during manufacture, and (3) analysis of outgoing finished products. Suppose some characteristic like tensile strength of the paper is to be controlled. The tensile strength will vary randomly and no action need be taken if the fluctuations are within a permissible range. Moreover, due to the continuous nature of the process, it may be expected that there is some degree of autocorrelation between successive observations. There is also a considerable time lag between a process adjustment and the response to the adjustment. The frequency of sampling is more or less determined by the characteristics of the process. For instance, a sample may be taken only when the paper sheet is cut for winding on a different reel. Model III may be used to describe this process. Due to the large time lag of response, any continuous adjusting action may not prove effective. When the process has been detected to have gone out of control, the assignable cause responsible for the increase in the variation of the process mean must be located and rectified. The assignable cause may pertain to a change in the quality of the raw material or a change in a physical characteristic of the manufacturing process. When the rectification has been performed, the process will revert back to its normal state of control.

Barnard (6) and Goldsmith and Whitfield (22) have studied Model III from a "Level Two" design viewpoint. Recently, Barish and Hauser (4) have published a paper in which a detailed study of Model III from a Level Three standpoint has been described. Barnard (6) has, for the first time, applied the theory of stochastic processes in the design of process control charts. He considers the use of cumulative sum charts in controlling time dependent stationary processes. He suggests that the primary function of the control chart should be to provide an estimate of the current process mean, together with a standard error or its equivalent, on the basis of which estimate, appropriate action may be taken to control the process.

Barnard suggests a model which consists of a "Poisson" process of "jumps" occurring at the rate of λ jumps per unit time. At each jump the process is shifted by an amount δ , which is itself a normally distributed variate with zero mean and variance σ_{δ}^{2}. A series of corrections $c(t_i)$ will be applied at various times t_i so that the resultant deviation of the process mean from the target value, at any time t will be

$$\Delta(t) = \sum_{t_i < t} \delta_{t_i} - \sum_{t_i < t} c(t_i)$$

Barnard assumes that the cost of departures from the target values will be proportional to the squared error

$$\frac{1}{T} \int_0^T \Delta^2(t)\,dt \;.$$

and obtains the best linear estimator of the process mean as a weighted mean of the recent observations with the weights proportional to the solutions of a set of linear equations. The weights decrease as one goes back in time in the series of observations. He also suggests that the enormous convenience of an exponential weighting would probably outweigh the slight loss of efficiency. The estimator would then be given by

$$X_n = [x_n + Ax_{n-1} + A^2x_{n-2} + \ . \ . \ .]/(1-A)$$
$$= \frac{1}{1-A}\, x_n + AX_{n-1}$$

where A is the smaller root of the quadratic

$$1 = A + \quad \lambda\sigma_\omega^2/(1-A).$$

As a rough graphical procedure, a parabolic mask may be used on the cumulative sum chart instead of the more usual V-mask. The parabola is placed with its vertex over the current point on the graph and the axis rotated so as to include the greatest possible number of consecutive points in the parabola, counting backwards from the current point.

Thus, the recent observations will have heavy weights. The final direction of the axis will represent the shift in the process mean from its target value.

Barnard then suggests procedures for the estimation of λ and σ_ω^2 from the past data. If $x_1, x_2, \ldots, x_n$ is a sequence of observations in which each x_i has the variance unity, then the mean square successive difference

$$D = \frac{1}{n-1} \sum_{i=1}^{n-1} (x_{i+1} - x_i)^2$$

has a mean $(2 + \lambda \sigma_\omega^2)$. Hence (D-2) can be taken as an unbiased estimator of $\lambda \sigma_\omega^2$ and it will then be sufficient to find an estimator of λ, for σ_ω to be determinable. There are various methods to estimate λ but all the methods suggested are valid only under limited circumstances. For instance, if $\sigma_\omega^2 \gg 1$, a simple count of the number r of sampling intervals of unit length in which a jump occurs out of a set of n such intervals is enough.

$$\hat{\lambda} = -\log_e(r/n)$$

Barnard concludes his article by suggesting that the field of stochastic control of processes has been relatively unexplored up to the present time and hence, there is need for substantial work to be done in the future.

When a process is considered to have gone out of control, one of two general types of decisions can be made: either investigate and, if necessary, locate and remove the assignable cause; or adjust the level of one or more input variables to compensate for the apparent changes in the output parameters. Barish and Hauser (4) consider only the latter category in their study of Model III from a Level Three design viewpoint. It is assumed that the number of arrivals of assignable causes of variation during any time interval is a Poisson variable with mean λ. For facilitating simulation, a unit time interval is defined as that length of time during which the probability of exactly one arrival is approximately 0.01. When an assignable cause occurs, the process mean is subject to "jumps", whose magnitude, δ_i, is a random variable with zero mean and variance $\theta^2 \sigma^2/n$. The general process is then represented by

$$X(t) = \mu + \sum_{i=1}^{t} \delta_i + \sum_{k=1}^{b} \sum_{j=1}^{t-a} \phi_{kj} \eta_{kj} + \epsilon_t$$

where

μ is the target process mean.

a is the time delay between a decision and the corresponding adjusting action.

b is the number of adjusting devices.

η_{kj} is the change introduced by adjusting device k at time j.

ϕ_{kj} depends on the interaction that the adjusting action η_{kj} has with the other adjusting actions and the current process level.

ϵ_t is the random variation of the process due to chance causes.

As related to Model III, this model assumes that $\rho_m = 0$ for all $m > 0$, $\theta_1 = 0$, and $\theta_2 = \theta\sqrt{n}$. If there is no time delay ($a = 0$), only one adjusting device ($b = 1$), and no interaction ($\phi_{kj} = 1$), the model reduces to

$$X(t) = \mu + \sum_{i=1}^{t} \delta_i + \sum_{j=1}^{t} \eta_j + \epsilon_t$$

This simplified model is investigated with the help of Monte Carlo simulation for various control chart designs with the objective of minimizing the total cost function consisting of the cost of defectives, the fixed and variable costs of sampling, cost of interpretation and decision making and cost of adjustment, all on a unit time basis. The interrelationship among decision rules, sample sizes and sampling intervals is examined. Five decision rules for detecting the out-of-control state of the process are used: 1-sigma, 2-sigma and 3-sigma control limits, run rules based on Western Electric control chart procedures, and geometric moving average rule as stated by Roberts (46). For each decision rule and each set of process parameter values, the sample size is varied, keeping the sample size-sampling interval ratio constant, until a minimum cost is reached. This process is repeated at different sample size-sampling interval ratios until the sample size and sampling interval, giving the minimum cost, are found. Five figures are given summarizing the results and giving the optimal decision rule, sample size and sampling interval for different combinations of process parameters. The process model and the range of parameter values are necessarily limited but the results are expected to be useful not only in processes represented by the model but also as first approximations in processes where some assumptions are violated.

The application of cumulative sum charts for the control of autocorrelated continuous processes seems to be extremely encouraging. Goldsmith and Whitfield (22) have evaluated the average run lengths of cusum charts for different process parameters by means of Monte Carlo simulation on a digital computer. The power of cusum charts is then compared with the power of conventional control charts. They also investigated the effect of serial correlation between observations with the help of an exponentially weighted autoregressive model. It is found that for large deviations of the process mean, the serial correlation has little effect on the ARL but when the magnitude of the critical shift in the process mean is small, positive correlation in the process increases the effectiveness of the cumulative sum control schemes (halving the ARL in some cases). On the other hand, negative correlation between successive observations leads to higher ARLs.

As the estimation of the parameters of the model may be a formidable task, it is relevant to describe some estimation procedures. When errors are autocorrelated, the straight-forward extension of standard estimation procedures for linear regression has been stated by Plackett (45). The resulting estimation method is valid provided that the number of parameters to be estimated from the data is small compared with the number of observations. If ω_t denotes the t'th residual, namely $\left\{X(t) - \mu(t)\right\}$,

$$E(\omega_t) = E\left\{X(t) - \mu(t)\right\} = 0$$

The autocovariance of time lag m is given by

$$\lambda_m = E(\omega_t \omega_{t-m}) = E\left[\left\{(X(t) - \mu(t)\right\}\left\{X(t-m) - \mu(t-m)\right\}\right].$$

λ_m can be estimated by

$$\hat{\lambda}_m = \frac{1}{N-m} \sum_{t=m+1}^{N} \omega_t \omega_{t-m} .$$

where N is the sample size of the past data. When the value of λ_m tends to zero as m increases, an integer m_o is found which is small compared to N and such that λ_m can be taken as zero for $m \geq m_o$. If such an integer exists, the values of λ_m for $m < m_o$ are directly substituted for the corresponding autocovariances to obtain

$$\theta_1{}^2 \sigma^2 + \frac{\sigma^2}{n} = \sum_{1-m_o}^{m_o-1} \hat{\lambda}_m$$

As σ can easily be estimated from the within sample variance, assuming independence within samples, θ_1 can also be determined.

If the autocorrelation coefficient ρ_m is approximately zero for all $m > 1$, the model is simplified considerably and a simple exponentially weighted predictor may be used. This assumption implies that the autocovariance of the stationary series does not extend beyond a lag of one unit of time, or, one sampling interval. The model can then be represented as

$$X(t) = \mu(t) + \alpha[X(t-1) - \mu(t-1)] + \nu_t + \epsilon_t$$

where

ν_t = a random variable statistically independent of t and normally distributed with zero mean and variance $\theta_1^2 \sigma^2$.

The value of α can be determined with the help of the assumption of the statistical independence of ν_t and t, or

$$E[\nu_t \nu_{t+1}] = 0 .$$

Consider a process with $\mu(t) = 0$ for all t, $\rho_1 = \rho$, and $\rho_m = 0$ for all $m > 1$. From the model described above

$$\begin{aligned} E[\nu_t \nu_{t+1}] &= E[(X_t - \alpha X_{t-1})(X_{t+1} - \alpha X_t)] \\ &= E(X_t X_{t+1}) - \alpha V(X_t) + \alpha^2 E(X_t X_{t-1}) \\ &= (1+\alpha^2)\rho V(X_t) - \alpha V(X_t) \\ &= V(X_t)[\rho(1+\alpha^2) - \alpha] \end{aligned}$$

This expectation is zero, if and only if $\rho(1+\alpha^2) - \alpha = 0$,

$$\text{or} \quad \alpha = \frac{1-\sqrt{1-4\rho^2}}{2\rho} \qquad (0 < \rho \leq 0.5) .$$

The variance of ($\nu_t + \epsilon_t$), $(\theta_1^2 \sigma^2 + \frac{\sigma^2}{n})$, as well as the within sample variance, σ^2, can be estimated by the analysis of variance of the past data. Hence, the parameter, θ_1, is easily determinable.

Box and Jenkins (8) consider the problem of adaptive optimization and quality control of continuous processes. It is often necessary to adjust some variable input, such as the concentration of consecutive batches of a product, to keep the output close to a specified target value. This is shown to be a problem of prediction, and the paper is devoted mainly to the estimation from past data of the "best" adjustments to be applied. Historical data in the form of a non-stationary stochastic, discrete time-series are considered and after discussing the optimal predictors they conclude that the theoretical approaches would be extremely difficult to apply in practice. The following model is proposed to be adequate for many practical purposes.

Let the time periods be denoted by the subscripts 0, 1, 2, . . . , p, p+1, . . .

$\triangle$ = the difference operator of the function f(p), i.e., f(p) - f(p-1).

Z_p = predicted response at period p.

e_p = error at period p = (actual response-predicted response) at period p.

γ_{-1} = first difference parameter.

γ_0 = proportional parameter.

γ_1 = cumulative parameter, equivalent to integration for a continuous time series.

Then the predicted change in the response, ΔZ_{p+1}, is given by

$$\Delta Z_{p+1} = \gamma_{-1}\Delta e_p + \gamma_o e_p + \gamma_1 \sum_{p_i \leq p} e_{p_i} .$$

The usefulness of this equation is suggested by the fact that the exponentially weighted predictor is a special case of this equation with the proportional term only.

$$\begin{aligned} \Delta Z_{p+1} &= \gamma_o e_p \\ \text{or, } Z_{p+1} - Z_p &= \gamma_o (X_p - Z_p) \\ \text{or, } \quad Z_{p+1} &= \gamma_o X_p + (1-\gamma_o) Z_p \end{aligned}$$

In the writer's opinion, the development of the Model III has not, as yet, received the attention it deserves. The present theory of stochastic processes should be extended to unify and further develop the design of control chart procedures from the viewpoint of this model.

CHAPTER VI

MODEL IV

$$X(t) = \mu(t) + \epsilon_t$$

where $X(t)$ = an estimate of the process mean, $\mu(t)$, based on n observations at time t.

$\mu(t)$ = an estimated or known function of time according to which the process mean behaves.

ϵ_t = random variation inherent in the production process and the measurement system. It is assumed to be statistically independent of t and normally distributed with zero mean and variance $\phi^2\sigma^2/n$.

$1 \leq \phi \leq 1 + \delta$, when the process is in control.

$1 + \delta < \phi$, when the process is out of control.

$\delta \geq 0$.

Model IV represents only those processes in which the random variation from item to item is of primary importance. The in-control and out-of-control states of the process are distinguished by the magnitude of the variance of the random variation denoted by $(\phi\sigma)^2$. When the process is in control, the standard deviation of the random variation of the process may fluctuate between σ and $(1 + \delta)\sigma$. When ϕ assumes a value greater than $(1 + \delta)$, the standard deviation of the process increases to a value greater than $(1 + \delta)\sigma$ and the process is assumed to have gone out of control. The explicit statement of δ sets the upper boundary to the permissible fluctuation of the process standard deviation.

The application of this model alone for the control of a process is relatively rare. In practice, usually, both the process mean and the variability are controlled. An illustration of controlling only the variability of manufactured items may pertain to the manufacture of gunsights. Finished gunsights are often tested only for variability to attain the necessary accuracy in use. Another example is the muzzle velocity of artillery projectiles. The variability in this characteristic is of prime importance, whereas variation in the mean level can be compensated in using the projectiles.

There are various conventional tools for the design of control charts for controlling the process variability. The σ-chart, s^2-chart, and the R-chart are the most important. When using the σ-chart, the average σ obtained from the past data is used in conjunction with formulas designed to give three-sigma control limits. Since σ is not normally distributed, the exact α risk associated with this control chart is difficult to ascertain. To avoid the above-mentioned difficulty, the s^2-chart is often used as a substitute. The quantity s^2 is an unbiased estimate of the population variance, and the probability limits for it can be readily derived from the Chi-Square distribution.

The most popular and widely used chart for controlling the process variability is the range chart, or the R chart. The reason for its popularity is the extreme simplicity in the computation of the range from a sample set of data. Patnaik (42) showed that the square of the average range has a distribution that is approximately of the form of a Chi-Square distribution. He has worked out the conversion factors d_2^* and the equivalent degrees of freedom associated with the range estimator for various values of the number and the size of the samples used in determining $\bar{R}$. The significance of this work is that in any analysis involving s, the estimator s can be replaced by the more readily computed $\bar{R}/d_2^*$.

On the basis of Pearson's studies (43,44), tables have been established for determining the three-sigma control limits for R charts. Also, the probability of obtaining values of R/σ less than specified values have been computed and tabulated for various sample sizes by Pearson (43). On the basis of these tables, Scheffe (47) and Duncan (13) give the operating characteristic curves for the range chart. Scheffe employs $B\sigma_R$ limits while Duncan has studied only $3\sigma_R$ limits.

Howell (24) has evaluated the control chart which calls for plotting the largest and the smallest observations as a device for detecting an increase in the random item to item variation. For samples of size three and five, the average number of samples required to detect a lack of control with probability 0.99 are given for values of ϕ from one to two. This evaluation indicates that the chart for the largest and the smallest values requires fewer samples than the R chart alone, but somewhat more than the $\bar{X}$ and R charts combined. Weiler (54) found that the use of run decision rules, which were found quite effective in controlling the process mean, are of little or no value in connection with R charts.

Johnson and Leone (26) have suggested the use of cumulative sum schemes for controlling the process variability. The cumulative sum scheme is shown to be an extension of the sequential probability ratio test. The equations for the acceptance boundaries for both one-sided and two-sided tests are derived from which the values of the necessary parameters of the cumulative sum chart are obtained.

Pearson (44) has investigated the merits of two possible approximations to the distribution of the sample range in samples of size n from a normal population, with standard deviation equal to one. The distribution is approximately either (a) c.(Chi with ν_1 degrees of freedom) or, alternatively, (b) c'. (Chi-square with ν_1' degrees of freedom) where c, c', ν_1, ν_1' depend on n. Pearson finds that (a) is the better approximation if n is less than ten, otherwise (b) is better. On the basis of this paper, Johnson and Leone (25) have investigated the use of sample ranges in cumulative sum schemes. Approximation (b) leads to the simpler form of cumulative sums, using only the sums of the sample ranges, while for (a), the sums of the squares of the sample ranges must be calculated. In either case, a cumulative sum chart can be conveniently constructed only if the sample size is constant since c, c', ν_1, ν_1' depend on n. The difference between the accuracies of the two approximations does not appear to be sufficiently great, compared with the other sources of variation likely to be encountered in applying the method, to warrant using the more complicated technique resulting from using (a).

Complete and exhaustive tables have been prepared by Johnson and Leone for determining the dimensions of the mask considering the scaling factors and different levels of α and β. The constants c, c', ν_1, ν_1', are also given for different sample sizes. From a comparison of average run lengths based on standard procedure and cumulative sum scheme, almost identical conclusions are derived as are obtained for controlling the process mean. The cumulative sum charts are superior when δ and α are small.

To control the variability of a process, Model IV describes the statistical procedure quite realistically and the parameters of this model can be estimated quite easily. As this model is adequate for a statistical description of the variability of a process, the conventional control chart procedures are often sufficient.

CHAPTER VII

MODEL V

Model V provides for the formation of a group of composite models by combining Model IV with Model I, II, or III.

In the first three models, the control procedures presume that only the process mean should be controlled while in Model IV, only the variability of the process has to be controlled. Usually, however, the control of both the process mean and the process variability are important as the two characteristics supplement each other in describing a process. The control chart for either the mean or the variability of the process is eliminated only if it is established beyond reasonable doubt that the "serious" variation of that aspect of the process is too rare or of too small a consequence to warrant the use of control charts.

Three studies of this model refer to the combination of Models I and IV. Duncan (13) gives the operating characteristic curves for an $\bar{X}$ chart when both the process mean and the standard deviation vary. On a figure with k plotted versus ϕ, Duncan gives the contours for the locus of the combinations of (k, ϕ) which give probabilities of 0.5 and 0.1 of accepting the null hypothesis for sample sizes of five and ten. The procedure for constructing other contours with different probability values is also given along with a few computed values of the operating characteristic function. From the contours drawn, it is concluded that the $\bar{X}$ chart does not give much protection against shifts in the process standard deviation and hence R charts are necessary. Moreover, as the allowable probability of type II error is decreased, the $\bar{X}$ chart gives considerably worse protection. Furthermore, within the range of practical application, the protection against changes in the process mean gets increasingly worse as the allowable probability of type II error is decreased and the process standard deviation increased.

Page (35) has considered the use of a single $\bar{X}$ chart with two sets of control limits for controlling both the process mean and the process variability. If the changes in the process standard deviation are relatively rare, this chart can eliminate the need for the R chart. The decision rule is as follows: choose λ, n; plot the means of samples of size n on a chart with both action and warning lines; take action if (i) any point falls outside the action lines, or (ii) λ consecutive points fall outside the warning lines, or (iii) two out of any set of λ consecutive points fall outside the opposite warning lines. This scheme cannot be very sensitive regarding the detection of any change in the process standard deviation and will become less sensitive with increase of the sample size. Moreover, it may often be misleading regarding the suggestion of the assignable cause of variation. If a point falls outside the action line due to a change in σ, or λ consecutive points fall outside the same warning line, it will give the impression that the process mean has changed. Page has given two short tables of the average run lengths for changes in the mean by $k\sigma$ units, when σ is constant, and for changes in σ from σ' to $\phi\sigma'$, when μ is constant. The final conclusion for the rule seems to be in favor of using moderate sample sizes.

Howell (24) has evaluated the joint variation of k and $\emptyset$ for sample sizes of three and five. He advocates the use of control charts for the largest and the smallest values in the sample and the results of his study indicate only a moderate loss of power in using these charts as compared to the conventional $\overline{X}$ and R charts over the complete range of the values of k and $\emptyset$ studied.

In using Model V, which provides for the combination of two models, the design of the control chart procedure will be principally based on the constituent model which describes the important process characteristic. For instance, if a combination of Model I and Model IV is used, and if the variability of the process seldom goes out of control, the composite control chart design will depend on the optimal control chart parameters suggested by Model I.

Although some work has been done, further research needs to be directed in formulating Model IV designs to satisfy the criterion of optimality of both the constituent models when the control of the process mean and the process standard deviation are equally important.

CHAPTER VIII

MODEL VI

Model VI provides for variations in Models I through V in regards to the assumptions about the distribution functions of the random variables involved in these models. In the first five models, the random variables are assumed to be normally distributed.

The assumption of normality, according to the Central Limit Theorem, is usually valid for the random variation ϵ_t provided the observations comprising a sample are independent, and the number of observations is greater than four or five. For the distribution of ω_t, however, the assumption of normality is merely one of convenience and logic. Due to the multiplicity of causes that may generate ω_t, a normal distribution may very well describe the random variate. If the past data fit a significantly different distribution better, the assumption of normality should be discarded.

Burrows (9), Gayen (20), and Gephart (21) have examined the behavior of control schemes when the production variable has a non-normal distribution. The studies of Burrows and Gephart are particularly important from the quality control viewpoint. Burrows (9) evaluates the average run lengths of $\bar{X}$ control schemes for certain asymmetric distributions. He then compares these average run lengths with those based on normal distributions. From the comparison, he concludes that quite moderate skewness in the parent distribution can lead to sizeable differences in the performance of a quality control scheme. To prove it on a theoretical basis, he shows that the r'th cumulant of the distribution of the mean of a sample of size n is a fraction $1/n^{r-1}$ of the r'th cumulant of the parent distribution. Hence, if γ_1 and γ_2 are the standardized cumulants for the skewness and kurtosis measurements of the parent distribution, those of the distribution of the sample mean are given by

$$\bar{\gamma}_1 = \frac{\bar{K}_3}{\bar{K}_2^{\,3/2}} = \frac{K_3}{K_2^{\,3/2} n^{1/2}} = \frac{\gamma_1}{\sqrt{n}} ,$$

and

$$\bar{\gamma}_2 = \frac{\bar{K}_4}{\bar{K}_2^{\,2}} = \frac{K_4}{K_2^{\,2} n} = \frac{\gamma_2}{n} , \text{ where } K_r = \text{r'th cumulant.}$$

Thus, for samples of size 4, the skewness of the mean is one-half that of the parent population and hence, for small sample sizes, the effect of skewness in the parent population on the skewness of the distribution of the sample means is fairly large.

Gephart (21) has studied the effects of non-normality on the sampling distribution of the range. Thirteen different parent populations with varying degrees of skewness and kurtosis were studied by drawing 2,000 random samples of sizes 3, 5, 7, and 10, from each of the 13 populations. The sampling distributions of the relative range, R/σ , and the moment constants of the 13 populations are compared with the theoretical values for the standardized normal population. From this comparison, it is concluded that if the control chart constants for the normal curve are used when the population is actually non-normal, the tendency is to produce tight control limits, or to increase α. The findings also imply that the increase in the probability of a type I error, α, would not be considered objectionable even for considerably high values of skewness and kurtosis as compared to the normal distribution values.

Cowden (11) suggests several methods for establishing control charts for non-normal distributions. If the sample size is increased, the skewness and the dispersion of the sample means are reduced and the distribution of the sample means approximates a normal distribution more closely. A simple method of obtaining nonsymmetrical control limits is based on the arbitrary assumption that the nonsymmetrical distribution consists of the left side of a normal distribution and the right side of another normal distribution, each distribution having the same mode and mean but a different standard deviation. This method is known as the split distribution method.

Sometimes control charts for means with nonsymmetrical control limits can be based on theoretical distributions on the assumption that the population conforms to a specified type of curve such as Gram-Charlier or Pearsonian Type III curve. The control limits can be established with the help of selected probability points which are given in tables. Sometimes the difficulties presented by extreme asymmetry can be overcome by logarithmic or some other transformation of the quality control data. Although this procedure involves the complexities of transformation of the data, once the function is established, the procedure is identical as for conventional charts.

No investigation has as yet been directed towards establishing the validity of using cumulative sum schemes for non-normal populations. Cumulative sum charts may be appropriate for asymmetric data, as the cumulations of sample means will progressively reduce the skewness and the dispersion of the plotted points due to the increase in the size of the cumulative sample.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

The task of selecting an optimal quality control procedure for a process would be simple if a set of selection rules, based on the relevant process characteristics, could be defined. In view of the wide diversity of processes and the limited growth of the optimizing techniques of quality control procedures, such a set of rules cannot be derived at present. The quality control engineer must exercise his judgment in the selection of the control chart plan, and possibly modify the plan, to take into account the peculiarities of the process he is striving to control. The chief objective of such a selection procedure is the minimization of the long run average net quality losses of the process.

Choice of a Model

A comprehensive set of six process quality models has been proposed in order to provide guidelines for the classification of different processes from the quality control viewpoint. Considerable logical reasoning and subjective judgment must be exercised to select a model to describe the statistical properties of a process. The following suggestions are made in order to aid this selection.

The first three models are used for controlling only the process mean while Model IV is selected for controlling only the natural variability of the process. When both the process mean and the process variability need to be controlled, one of the first three models can be combined with the fourth model to yield Model V. Model V, therefore, is not an independent model, but consists of the various possible combinations among the first four models. Hence, when the fifth model is used, the parent models should be studied to yield the optimal control chart procedure. Model VI should be used for processes where the random variables in the process are known to be non-normal.

Model I

Model I describes a process whose mean is to be controlled and which satisfies either of the following conditions. First, the process mean is subject to random fluctuations from time to time and while the behavior of the fluctuations is unknown, the magnitude of a "serious" shift in the process mean is known. Secondly, this model may be used for simplicity where the situation does not warrant the consideration of the more elaborate Models II and III.

The location and rectification of the assignable causes of variation in a Model I process is usually done manually as opposed to the continuous adjustments of processes performed with the help of feedback loops for which a more precise description of the process behavior is necessary. Although the first model often leads to a comparatively low precision in the prediction of the process behavior, the simplicity and convenience of use may outweigh its disadvantages.

Model II

Model II is identical to the first model except only in the assumption of the behavior of the process mean when the process is running out of control. If the behavior of the process mean, in both out-of-control and in-control states, can be determined and if the behavior is a determinable function of time, the second model should be used.

This model is suitable for processes which require periodic maintenance, for instance, periodic sharpening of the tool in a metal cutting operation or replenishing the supply of a catalyst in a chemical process. Model II is also appropriate for processes which use automatic controlling devices because the control chart forms a part of the feedback loop in signalling not only when the adjustments are necessary, but also the magnitude of the adjustments.

Model III

In many industries, the process behavior cannot be represented as a deterministic function of time. The process mean behaves in a random manner regarding both the magnitude of deviations from the target value and in the occurrence of such deviations. The successive observations of the process mean may form a time dependent series of observations as distinct from the conventional quality control concept of a series of independent observations. Model III describes such a time dependent, random behavior of a process.

Model IV

If only the random variation from item to item is to be controlled, the process is uniquely described by Model IV. The selection of this model, therefore, does not entail any subjective judgment except for the preliminary decision to control only the process variability and the assumption of normality of the random variable.

Model V

This model represents the group of composite models which can be formed by combining Model IV with Models I through III. The two models comprising Model V are selected beforehand on the basis of the process characteristics, and hence the formulation of Model V does not involve any difficult selection procedure.

Model VI

When the random variables in one of the first five models are known to be non-normal, this model is used as a modification of the relevant model. Often, however, the non-normality of the data is ignored if the loss in precision by applying the control chart procedures based on normal distribution is not significant.

Use of Models

Model I

In Model I, where a shift of $k\sigma$ units in the process mean is considered serious, the constant, k, is the controlling factor in the design of the optimal quality control chart procedure. Four different ranges of the value of k will be considered for the purpose of control chart design.

$k < 0.5$. The use of control chart to detect a shift in the process mean of magnitude less than 0.5σ is relatively rare. If the specification limits are sufficiently close with respect to the process capability to warrant such tight control limits, one should consider redesigning the process to lessen the process variability. If this is not economical, 100 per cent inspection may be required to supplement the control chart. The cumulative sum chart is the most powerful quality control test to detect a shift in the process mean of magnitude less than 0.5σ. If the shift is not sustained for at least four or five sampling intervals, control charts with warning lines may be helpful. Conventional control charts will be seldom practical in detecting such small shifts in the process mean due to the extremely large sample size and the tight control limits necessary.

$\underline{0.5 \leq k \leq 1.5}$. The use of conventional sample sizes with 3-sigma control limits is of little value in detecting a significant shift in the process mean. Under these circumstances, one of the following three alternative quality control chart procedures may be used. The conditions of applicability of each procedure are stated along with each procedure.

(i) Use conventional control charts with large sample sizes. Sample sizes of 8 to 30 should be used with a large sampling interval. If the cost of looking for assignable causes of variation is large, comparatively wide control limits (3 or 3.5 sigma limits) should be used, while for low cost of looking for trouble, narrower control limits (2 or 2.5 sigma limits) should be used. The conventional control charts should be applied only if the following conditions are satisfied. First, the unit cost of inspection and charting is relatively small. Secondly, the loss due to the production of a defective item is comparatively small. Thirdly, the production rate is low and fourthly, a simple control chart procedure is required. If any of these conditions is violated, conventional control charts should be discarded in favor of either of the following two charts.

(ii) Use cumulative sum chart. If the shift in the process mean is sudden and is sustained for at least two or three sampling intervals, cusum charts are appropriate. These charts are of great value, especially if the complexity of charting is not objectionable and the unit cost of inspection is comparatively high. When a shift in the process mean occurs, the cumulative sum chart will give a rapid indication of the change. The sample size should be moderate and can be determined, along with the sampling frequency, from the tables given by Page (38) and Johnson and Leone (26). The visual detection power of the cusum chart regarding the point of initiation of the change may be very helpful. The production subsequent to that estimated point can be fully screened to eliminate the greater number of defective units produced after the inception of the assignable cause.

(iii) Use control charts with warning lines. Sometimes neither of the above schemes may be applicable; for instance, it may be impractical to take the large samples as suggested in (i) above, and the shift in the process mean may not be sustained at one level long enough for the efficient use of cusum charts. In such a situation, control charts with warning lines should be used. The use of warning lines in conjunction with conventional control limits can eliminate the need for large sample sizes. Various run rules are given by Page (35) and Weiler (53,54) along with the necessary tables for determining the control limits, the warning line limits, and the sample size. Control charts with warning lines are suitable when moderate complexity in charting can be tolerated and the unit cost of inspection is not too great.

<u>$1.5 < k \leq 2.5$.</u> The selection of the control chart procedure for detecting shifts of 1.5σ to 2.5σ units in the process mean depends primarily on the important economic factors. The powers of the three tests — the conventional charts, the conventional charts with warning lines and the cumulative sum charts, are not markedly different in this range. If the tests are inexpensive and simplicity of charting procedure is required, conventional charts should be used. The selection of the sample size and the placement of control limits can be derived from the tables given by Duncan (12) and Page (33). The optimal sample size will vary between 3 and 8. If visual estimation of the magnitude of the change as well as the point of initiation of the change is required, cumulative sum schemes will be more valuable. If moderate complexity in the charting procedure is not objectionable and only small samples can be taken, the run rules along with the conventional control limits may be used.

<u>$k > 2.5$.</u> When the detection of a shift in the process mean of more than 2.5σ units is important, the conventional control charts are preferable. The optimal sample size will be below five and hence there is very little disadvantage associated with the use of standard charts.

When conventional charts are used for any value of k, the following recommendations are stated by Duncan (12). The loss caused by the production of a defective item has its dominant effect on the sampling interval. If the unit loss is small, the sampling interval should be large, and vice versa. A large cost of looking for assignable causes of variation suggests the use of wide control limits. The unit cost of inspection and charting affects all the three parameters of the control chart — the sample size, the sampling frequency, and the placement of the control limits. For high unit cost of inspection, the optimum design calls for taking small samples, at large intervals between samples, and with narrow control limits. Comprehensive tables are given by Duncan for the selection of the optimum control chart parameters for different process and cost parameters.

It should be borne in mind that no single recommendation regarding the choice of the sample size, the frequency of sampling, and the placement of the control limits can be optimal for different values of process or economic characteristics. For a particular set of parameter values, the optimal values of the control chart parameters can be obtained by an analysis of the relevant factors. The analysis should be performed only in sufficient detail to yield the best results regarding both the costs of analysis and the marginal savings expected therefrom.

Model II

No comprehensive study of this model has been made up to date and hence there is a dearth of information on which the design of control chart procedure could be based. It is doubtful if the standard control charts are of much value in this model.

When the process goes out of control, the behavior of the process mean will be somewhat reflected in runs of plotted points and hence, run rules may perhaps be valuable. The advantage of using run rules may also be obtained by using either arithmetic moving average charts or geometric moving average charts. The information from more than one sample should be relatively more helpful in detecting the present state of the process. The weights assigned to the successive observations can be determined from historical knowledge of the behavior of the process in both out-of-control and in-control states. The sampling interval should be so chosen as to insure that at least one sample is taken from an out-of-control process before its mean value has reached the level where the probability of producing a defective unit becomes appreciable. The sampling interval will depend on the function $f(t)$, the specification limits and the process variability. For a linear $f(t)$, an example has been suggested in the illustration of the Second Level of Sophistication in Chapter II.

Cumulative sum charts will be of little value in controlling a Model II process. Truax (49,50) has found from practical experience that cusum charts should not be used when the process mean "drifts" gradually. Considerable research needs to be done on this model to establish some valid control chart procedures.

Model III

In processes which behave according to Model III, and where the use of a control chart is justified, the process parameters, λ and $(\theta_2 - \theta_1)$, influence the design of the control chart considerably. The magnitude of the assignable cause that it is important to detect is given by $(\theta_2 - \theta_1)$ and the arrival rate of assignable causes by λ. For large $(\theta_2 - \theta_1)$, the differentiation between chance causes and assignable causes is comparatively easy. λ , on the other hand, influences the sampling interval; the larger the rate of arrival of assignable causes, the shorter should be the sampling interval.

When a process behaving according to Model III is considered to have gone out of control, one of two general types of actions can be taken: either adjust the level of one or more input variables to compensate for the apparent changes in the output parameters; or, investigate and, if necessary, locate and remove the assignable cause. The former situation is applicable to processes using automatic controlling devices and when the time lag between an adjusting action and its response is very small. Moreover, the assignable causes should be such that an adjustment in an input variable can bring the process back into a state of control.

Barish and Hauser (4) have conducted a study in the behavior of a process as mentioned above. They conclude that, for such a process, the standard 3-sigma control limits are of little value. If $(\theta_2 - \theta_1)$ is very small, narrow control limits, like 1-sigma limits, may be used but otherwise, decision rules which use information from more than one sample, like geometric moving averages or run rules will be of great value. In the opinion of the writer, this conclusion should not be valid unless the successive observations are largely autocorrelated and the autoregression extends up to a large time lag. According to Barish and Hauser, the sample size and the frequency of sampling seem to be highly dependent on $(\theta_2 - \theta_1)$. Both the sampling interval and the sample size are large for small values of $(\theta_2 - \theta_1)$, and as $(\theta_2 - \theta_1)$ increases, the economic sample size decreases and the economic sampling interval shortens. From the tables given, it appears that the economic sample size is always below the conventional sample size of four or five.

In a usual Model III situation, however, the assignable cause has to be located and removed to bring the process back into control. Standard control charts, employing the usual constant A_2, should not be used as they will provide tight control limits. A range chart may possibly be used where the range is the value of the difference between successive observed process means. If the successive observations are autocorrelated, the $R_{\bar{x}}$ may denote the difference between observed process means m sampling intervals apart. Considerable research has to be done on $R_{\bar{x}}$ charts in order to establish their validity. If the successive observations of the process mean are significantly and positively autocorrelated, cumulative sum charts may be of considerable value. If the observations are relatively independent of one another, it is doubtful if the use of cusum charts will lead to quick detection of an assignable cause. For negative autocorrelation, cusum charts should not be used. For autoregressive processes, if a moving sum chart is used, the setting of the control limits and the center line may be based on the theorem by Hoeffding and Robbins (17). This theorem and its modification are stated subsequently on page 89. The plotted points will be given by $\sum_{i}^{m} \bar{x}_i / \sqrt{n'}$, where n' represents the number of sampling intervals considered for the moving sum, and the control limits will be given by

$$\sqrt{n'}\ \bar{\bar{X}} \pm 3\sigma\sqrt{1 + 2\sum_{1}^{n'} \rho_i}$$

where ρ_i is the autocorrelation coefficient of time lag i. Although interest in Model III has increased in the more recent quality control literature, considerable research has yet to be performed in order to establish the validity and usefulness of any existent or proposed control chart procedure.

Model IV

This is the only model that describes processes whose item-to-item variation is to be controlled. The various procedures, for example the s^2 chart or the R chart, to control such a process have been mentioned in Chapter VI. The range chart seems to be quite adequate for the usual processes. The slight increase in efficiency in using s^2 chart does not often justify the increase in complexity in the charting techniques. The efficiency of the estimate $\bar{R}/d_2$ of the process standard deviation decreases rapidly as the sample size becomes large. For large samples, therefore, the data should be broken down into sub-samples, each of size four or five, for each of which the range is calculated. These ranges are averaged to give the $\bar{R}$ chart. When δ is small, cumulative sum schemes may be used for increasing the efficiency of the control chart procedure. The usual value of δ, however, is large enough to render the added complexity of using cusum charts unnecessary.

In the opinion of the writer, the conventional quality control tools are adequate for controlling the process variability.

Model V

Usually the process mean and the process variability are studied together by the use of conventional $\bar{X}$ and R charts and seldom only one aspect of the process is controlled. When Model V is used to describe a process, the optimal selection of the sample size and the sampling frequency will be determined by the constituent model which describes the more important process characteristic. No satisfactory scheme has yet been proposed to incorporate both the charts for the process mean and the variability into one chart. If such a chart can be developed, the quality control procedure for a Model V process will be substantially simplified.

Model VI

It is very important to study the effectiveness of various models under conditions when the assumed normality of the distribution of the random variables is violated. If the past data has a distribution which is markedly skew, various methods, as suggested in Chapter VIII, may be used to establish a valid quality control chart procedure. If the past data are not adequate to establish the nature of the distribution, and if there is reason to believe that some of the basic assumptions of the normal distribution are not satisfied, the choice of the quality control procedure becomes more difficult.

One of the basic assumptions of the Central Limit Theorems of Kolmogorov, and Lindeberg and Levy, is the independence of the random variables comprising the sample. Hence, to establish the normality of the distribution of sample means, the observations constituting a sample should be independent. In practice, however, the sample observations are often taken successively, or very close together, to estimate the process mean and the process variability at the time of taking of the sample. It may be logical that such successive observations may not be independent and errors will be introduced in the subsequent quality control procedure. In such a situation, the Central Limit Theorem for dependent variables may be used.

Such a procedure may be based on the notion of m-dependence. The sequence of random variables $x_1, x_2, \ldots$ is m-dependent if $(x_1, x_2, \ldots, x_r)$ is always independent of $(x_s, x_{s+1}, \ldots)$ provided $s-r > m$. The sequence is called stationary if the joint distribution of $x_i, x_{i+1}, \ldots, x_{i+r}$ is independent of i for all r. Then, according to Hoeffding and Robbins (17), for the m-dependent stationary sequence with $E(x_1) = \mu$ and $E(|x_1|^3)$ existing, as $n \rightarrow \infty$, the limiting distribution of $\sum_1^n x_i / \sqrt{n}$ is normal with mean $n^{1/2}\mu$ and variance

$$V = \text{var}\ (x_1^2) + 2\ [\text{cov}(x_1x_2) + \ldots + \text{cov}(x_1x_{m+1})]$$

If $\text{var}(x_1^2) = \sigma^2$ and $\text{cov}(x_1x_{i+1}) = \rho_i \sigma^2$,

$$V = \sigma^2 + 2\sigma^2\ (\sum_1^m \rho_i) = \sigma^2(1 + 2\sum_1^m \rho_i)$$

This theorem is applicable when independence of individual observations within a sample cannot be assumed. It is logical to assume that independence will exist among the sets of observations separated by one or more sampling intervals. The cumulative sum schemes will be extremely valuable in such cases, where, as the number of samples taken increases, the Hoeffding and Robbins theorem will be increasingly valid, and the normal theory will apply for the limiting distribution of $\sum_1^n x_i / \sqrt{n}$. Substantial study needs to be done along this approach to compare the cumulative sum schemes with the conventional charts when the observations in a sample are not independent.

Conclusions

The six process quality models developed here together with the review of the pertinent literature is designed to point out the potentialities and procedures for Level Two and Level Three control chart design work. Although a great deal can certainly be accomplished by this approach to the design of control chart procedures, one should not lose sight of the fact that this analysis is based on manipulation of mathematical models in the "symbolic world". The recommendations based upon such an analysis will only be of value if the symbolic or the mathematical models adequately portray the physical situation in the "real world".

The survey of the literature indicates that additional research is needed on some of the process quality models, especially Model II, III, and VI. There is also an urgent need for the consideration of the economic factors in making Level Three designs. As the manipulation of these models requires rather high level talents in mathematics and statistics, there is also an ever growing need to translate these models into simplified rules to make them more readily applicable in industry.

BIBLIOGRAPHY

1. Aroian, L. A. and Levine, H., "The Effectiveness of Quality Control Charts", _Journal of the American Statistical Association_, _45_, (1950), 520-529.

2. Aroian, L. A., "What Makes a Quality Control Chart Tick", _Industrial Quality Control_, _10_, No. 11, (May 1954), 38-43.

3. Armitage, P., "Sequential Analysis with More Than Two Alternative Hypotheses and Its Relation to Discriminant Analysis", _Journal of the Royal Statistical Society_, _B12_, (1950), 137-144.

4. Barish, N. N. and Hauser, N., "Economic Design for Control Decisions", _Journal of Industrial Engineering_, _14_, (1963), 125-134.

5. Barnard, G. A., "Sequential Tests in Industrial Statistics", _Supplement to the Journal of Royal Statistical Society_, _B8_, (1946), 1-21.

6. Barnard, G. A., "Control Charts and Stochastic Processes", _Journal of Royal Statistical Society_, _B21_, (1959), 239-271.

7. Bowker, A. H. and Lieberman, G. L., _Engineering Statistics_, Englewood Cliffs, N. J.: Prentice Hall, Inc., (1960).

8. Box, G. E. P. and Jenkins, G. M., "Some Statistical Aspects of Adaptive Optimization and Control", _Journal of Royal Statistical Society_, _B24_, (1962), 297-332.

9. Burrows, P.M., "$\bar{X}$ Control Schemes for a Production Variable with Skewed Distribution", _The Statistician_, _12_, No. 4, (1962), 296-312.

10. Cohen, A. C., "The Frequency of Selecting Samples for Attribute Testing", _Industrial Quality Control_, _8_, No. 1, (July 1951), 32-34.

11. Cowden, D. J., _Statistical Methods in Quality Control_, Englewood Cliffs, N. J.: Prentice Hall, Inc., (1957).

12. Duncan, A. J., "Economic Design of $\bar{X}$ Charts Used to Maintain Current Control of a Process", _Journal of American Statistical Association_, _51_, (1956), 228-242.

13. Duncan, A. J., Quality Control and Industrial Statistics, Richard D. Irwin, Inc., (1959).

14. Ewan, W. D. and Kemp, K. W., "Sampling Inspection of Continuous Processes with no Autocorrelation Between Successive Results", Biometrika, 47, (1960), 363-380.

15. Ewan, W. D., "When and How to Use Cusum Charts", Technometrics, 5, No. 1, (1963), 1-22.

16. Feigenbaum, A. V., Total Quality Control, McGraw-Hill Book Co., Inc., (1961).

17. Fraser, D. A. S., Nonparametric Methods in Statistics, John Wiley & Sons, Inc., (1957).

18. Fraser, D. A. S., Statistics, an Introduction, John Wiley & Sons, Inc., (1958).

19. Freund, R. A., "Graphical Process Control", Industrial Quality Control, 18, No. 7, (January 1962), 15-22.

20. Gayen, A. K., "On Setting up Control Charts for Non-Normal Samples", Indian Statistical Quality Control Bulletin, 1, No. 1, (October 1953), 43-47.

21. Gephart, L. S., "A Study of the Effect of Non-Normality on Sampling Distribution of the Range", Wright Air Development Center Technical Report 55-128, (1955), 154 pp.

22. Goldsmith, P. L. and Whitfield, H., "Average Run Lengths in Cumulative Chart Quality Control Schemes", Technometrics, 3, No. 1, (1961), 11-20.

23. Grant, E. L., Statistical Quality Control, 2d.ed., McGraw-Hill Book Co., Inc., (1952).

24. Howell, J. M., "Control Charts for Largest and Smallest Values", Annals of Mathematical Statistics, 20, (1949), 305-309. Errata - op cit, 21, (1950), 615-616.

25. Ito, S., "On the Characteristics of Control Charts", Statistical Quality Control, 7, (May 1956), 299-303.

26. Johnson, N. L. and Leone, F. C., "Cumulative Sum Control Charts", in 3 parts, Industrial Quality Control, Part I, 18, No. 12, (June 1962), 15-21, Part II, 19, No. 1, (July 1962), 29-36, Part III, 19, No. 2, (August 1962), 22-28.

27. Juran, J. M., (ed.) Quality Control Handbook, 2d.ed., McGraw-Hill Book Co., Inc., (1962).

28. Kemp, K. W., "The Average Run Length of the Cumulative Sum Chart When a V-Mask is Used", Journal of Royal Statistical Society, B23, (1961), 149-153.

29. Kemp, K. W., "The Use of Cumulative Sums for Sampling Inspection Schemes", Applied Statistics, 11, (1962), 16-31.

30. King, E. P., "The Operating Characteristics of the Control Chart for Sample Means", Annals of Mathematical Statistics, 23, (1952), 384-395.

31. Mitten, L. G. and Sanoh, A., "The $\bar{\bar{X}}$ Warning Limit Chart", Industrial Quality Control, 18, No. 2, (August 1961), 15-19.

32. Moore, P. G., "Some Properties of Runs in Quality Control Procedures", Biometrika, 45, (1958), 89-95.

33. Page, E. S., "Control Charts for the Mean of a Normal Population", Journal of Royal Statistical Society, B16, (1954), 131-135.

34. Page, E. S., "Continuous Inspection Schemes", Biometrika, 41, (1954), 100-115.

35. Page, E. S., "Control Charts with Warning Lines", Biometrika, 42, (1955), 243-257.

36. Page, E. S., "A Test for a Change in Parameter Occurring at an Unknown Point", Biometrika, 42, (1955), 523-527.

37. Page, E. S., "On Problems in Which a Change in Parameter Occurs at an Unknown Point", Biometrika, 44, (1957), 248-252.

38. Page, E. S., "Cumulative Sum Charts", Technometrics, 3, No. 1, (1961), 1-10.

39. Page, E. S., "Cumulative Sum Schemes Using Gaging", Technometrics, 4, No. 1, (1962), 97-109.

40. Page, E. S., "A Modified Control Chart with Warning Limits", Biometrika, 49, (1962), 171-176.

41. Parzen, E., Stochastic Processes, Halden-Day, Inc., (1962).

42. Patnaik, P. B., "The Use of Mean Range in Statistical Tests", Biometrika, 37, (1950), 78-87.

43. Pearson, E. S., "The Probability Integral of the Range in Samples of n Observations from a Normal Population", Biometrika, 32, (1942), 301-310.

44. Pearson, E. S., "Comparison of Two Approximations to the Distribution of Range in Small Samples from Normal Populations", Biometrika, 39, (1952), 130-136.

45. Plackett, R. L., Principles of Regression Analysis, Oxford University Press, (1960).

46. Roberts, S. W., "Control Chart Tests Based on Geometric Moving Averages", Technometrics, 1, No. 3, (1959), 239-250.

47. Scheffe, H., "Operating Characteristics of Average and Range Charts", Industrial Quality Control, 5, No. 6, (May 1949), 13-18.

48. Shewhart, W. A., Economic Control of Quality of Manufactured Product, D. Van Nostrand Co., Inc., (1931).

49. Truax, H. M., "An Introduction to Cumulative Sum Charts", 15th Annual Convention Transactions, American Society of Quality Control, Philadelphia, (June 5-7, 1961), 103-112.

50. Truax, H. M., "Cumulative Sum Charts and Their Application to the Chemical Industry", Industrial Quality Control, 18, No. 6, (December 1961), 18-25.

51. Wald, A., Sequential Analysis, John Wiley & Sons, Inc., (1953).

52. Weiler, G. H., "On the Most Economic Sample Size for Controlling the Mean of a Population", Annals of Mathematical Statistics, 23, (1952), 247-254.

53. Weiler, G. H., "The Use of Runs to Control the Mean in Quality Control", Journal of American Statistical Association, 48, (1953), 816-825.

54. Weiler, G. H., "New Type of Control Chart Limits for Means, Ranges and Sequential Runs", Journal of American Statistical Association, 49, (1954), 298-314.

55. Wolfowitz, J., "On the Theory of Runs with Some Application to Quality Control", Annals of Mathematical Statistics, 14, (1943), 280-288.